Exercise on Integration

1.1 Substitution

Use a suitable substitution to evaluate the following integral.

1. $\int \frac{dx}{\sqrt{2-5x}}$
2. $\int \frac{e^{3x}+1}{e^x+1}\,dx$
3. $\int \frac{x}{\sqrt{1-x^2}}\,dx$
4. $\int x^2\sqrt[3]{1+x^3}\,dx$
5. $\int \frac{xdx}{(1+x^2)^2}$
6. $\int \frac{dx}{\sqrt{x}(1+x)}$
7. $\int \frac{1}{x^2}\sin\frac{1}{x}\,dx$
8. $\int xe^{-x^2}dx$
9. $\int \frac{(\ln x)^2}{x}dx$
10. $\int \frac{e^x dx}{2+e^x}$
11. $\int \frac{dx}{e^x+e^{-x}}$
12. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$
13. $\int \tan x dx$
14. $\int \frac{dx}{1+e^x}$
15. $\int x(x^2+2)^{99}dx$
16. $\int \frac{x}{\sqrt{25-x^2}}dx$
17. $\int \frac{x}{\sqrt{3x^2+1}}dx$
18. $\int \frac{x^2}{\sqrt{9-x^3}}dx$
19. $\int x(x+2)^{99}dx$
20. $\int \frac{xdx}{\sqrt{4x+5}}$
21. $\int x\sqrt{x-1}dx$
22. $\int (x+2)\sqrt{x-1}dx$
23. $\int \frac{xdx}{\sqrt{x+9}}$
24. $\int x^3(1+3x^2)^{\frac{1}{2}}dx$

1.2 Integration By Parts

1. $\int \ln x dx$
2. $\int x^2 \ln x dx$
3. $\int \left(\frac{\ln x}{x}\right)^2 dx$
4. $\int xe^{-x}dx$

5. $\int x^2 e^{-2x} dx$
6. $\int x \cos x dx$
7. $\int x^2 \sin 2x dx$
8. $\int (\ln x)^2 dx$
9. $\int \sin^{-1} x dx$
10. $\int x \tan^{-1} x dx$
11. $\int \ln(x + \sqrt{1+x^2}) dx$
12. $\int x \sin^2 x dx$
13. $\int \sin(\ln x) dx$
14. $\int x \sin 4x dx$
15. $\int x \cos^{-1} x dx$
16. $\int \tan^{-1} x dx$
17. $\int x^{99} \ln x dx$
18. $\int \frac{\ln x}{x^{101}} dx$
19. $\int x \sec^2 x dx$
20. $\int e^{2x} \cos 3x dx$

1.3 Reduction Formula

Prove the following reduction formulas.

1. $I_n = \int x^n e^{ax} dx$; $I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$, $n \geq 1$
2. $I_n = \int \cos^n x dx$; $I_n = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}$, $n \geq 2$
3. $I_n = \int \frac{1}{\sin^n x} dx$; $I_n = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1} I_{n-2}$, $n \geq 2$
4. $I_n = \int x^n \cos x dx$; $I_n = x^n \sin x + n x^{n-1} \cos x - n(n-1) I_{n-2}$, $n \geq 2$
5. $I_n = \int \frac{dx}{(x^2 - a^2)^n}$; $I_n = -\frac{x}{2a^2(n-1)(x^2-a^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)} I_{n-1}$, $n \geq 1$
6. $I_n = \int \frac{x^n dx}{\sqrt{x+a}}$; $I_n = \frac{2x^n \sqrt{x+a}}{2n+1} - \frac{2an}{2n+1} I_{n-1}$, $n \geq 1$
7. $I_n = \int (\ln x)^n dx$; $I_n = x(\ln x)^n - n I_{n-1}$, $n \geq 1$.
8. $I_n = \int_0^1 x^n \sqrt{1-x} dx$; $I_n = \frac{2n}{2n-3} I_{n-1}, n \geq 2$.

1.4 Trigonometric Integrals

Evaluate

1. $\int \frac{dx}{1-\cos x}$
2. $\int \sin^5 x \cos x dx$
3. $\int \sin 3x \sin 5x dx$
4. $\int \cos \frac{x}{2} \cos \frac{x}{3}\, dx$
5. $\int \cos^3 x dx$
6. $\int \sin^4 x dx$
7. $\int \sec^2 x \tan^2 x dx$
8. $\int \sec x \tan^3 x dx$
9. $\int \cot^2 x dx$
10. $\int \frac{dx}{\cos x \sin^2 x}$
11. $\int \frac{\sin x \cos^3 x}{1+\cos^2 x}, dx$
12. $\int \tan^5 x dx$
13. $\int \frac{dx}{\sin^4 x \cos^4 x}, dx$
14. $\int \sin 5x \cos x dx$
15. $\int \cos x \cos 2x \cos 3x dx$
16. $\int \cos^5 x \sin^3 x dx$
17. $\int \cos^5 x \sin^4 x dx$
18. $\int \sin^2 x \cos^4 x dx$

1.5 Trigonometric Substitution

Evaluate the following integrals by trigonometric substitution.

1. $\int \frac{x^2}{1+x^2}\, dx$
2. $\int \frac{dx}{(1-x^2)^{\frac{3}{2}}}$
3. $\int \sqrt{\frac{1+x}{1-x}}\, dx$
4. $\int \frac{dx}{(1+x^2)^{\frac{3}{2}}}$
5. $\int \frac{x^2 dx}{\sqrt{9-x^2}}$
6. $\int \frac{dx}{\sqrt{4+x^2}}$
7. $\int x^2\sqrt{16-x^2} dx$
8. $\int \frac{dx}{x^2\sqrt{x^2+4}}$
9. $\int \frac{dx}{(4x^2+1)^{3/2}}$
10. $\int \frac{1}{(2x-x^2)^{3/2}}$

1.6 Rational Functions

Evaluate the following integrals of rational functions.

1. $\int \frac{x^2 dx}{1-x^2}$
2. $\int \frac{x^3}{3+x}\, dx$
3. $\int \frac{(1+x)^2}{1+x^2}\, dx$
4. $\int \frac{dx}{x^2+2x-3}$
5. $\int \frac{dx}{(x^2-2)(x^2+3)}$
6. $\int \frac{x^2+1}{(x+1)^2(x-1)}, dx$
7. $\int \frac{x^2}{(x^2-3x+2)^2}, dx$
8. $\int \frac{x^2+5x+4}{x^4+5x^2+4}, dx$
9. $\int \frac{dx}{(x+1)(x^2+1)}$
10. $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\, dx$
11. $\int \frac{4-2x}{(x^2+1)(x-1)^2}\, dx$
12. $\int \frac{dx}{x(x^2+1)^2}$
13. $\int \frac{x^2 dx}{(x-1)(x-2)(x-3)}$
14. $\int \frac{x dx}{x^2(x^2-2x+2)}$

1.7 t-method

Use t-substitution to evaluate the following integrals.

1. $\int \frac{dx}{\sin^3 x}$
2. $\int \frac{dx}{1+\sin x}$
3. $\int \frac{dx}{\sin x \cos^4 x}$
4. $\int \frac{dx}{2+\sin x}$
5. $\int \frac{1-\cos x}{3+\cos x} dx$
6. $\int \frac{\cos x+1}{\sin x+\cos x} dx$

1.8 Miscellaneous

Evaluate the following integrals.

1. $\int \frac{(\ln x)^2}{x} dx$
2. $\int x(\ln x)^2 dx$
3. $\int \frac{x dx}{\sqrt{1-x^2}}$
4. $\int \frac{x+4}{(x+1)^2} dx$
5. $\int \frac{\cos^3 x}{\sin^2 x} dx$
6. $\int \frac{x dx}{(1+x^2)^2}$

7. $\int \frac{e^{2x}dx}{1+e^x}$
8. $\int \frac{dx}{x(1+2\ln x)}$
9. $\int \cos^2 x \sin^3 x dx$
10. $\int \frac{\sin 2x}{1+\cos^2 x}dx$
11. $\int \frac{e^{\frac{1}{x}}}{x^2}dx$
12. $\int \frac{\sin x}{\cos^2 x}dx$
13. $\int x\tan^2 x dx$
14. $\int \frac{\cot x}{1+\sin x}dx$
15. $\int \frac{x^3dx}{x^2-1}$
16. $\int \frac{dx}{e^{2x}+e^x-2}$
17. $\int \frac{\ln x}{x\sqrt{1+\ln x}}dx$
18. $\int \frac{\sqrt{9-x^2}}{x^2}dx$
19. $\int \frac{x^2dx}{x^2+1}$
20. $\int \frac{dx}{\sqrt{x^2+9}}$
21. $\int \frac{\cos^3 x}{\sin x}dx$
22. $\int \frac{x^2+8}{x^2-5x+6}dx$
23. $\int \frac{xdx}{\sqrt{x-2}}$
24. $\int \frac{dx}{\sqrt{1+e^x}}$
25. $\int \cos(\ln x)dx$
26. $\int x\sin^2 x dx$
27. $\int \frac{dx}{\sqrt{e^x-1}}$
28. $\int \frac{4dx}{x^2\sqrt{4-x^2}}$
29. $\int \frac{x+1}{x^2(x-1)}dx$
30. $\int \sec^3 x\tan x dx$
31. $\int x^3\sqrt{x^2+1}dx$
32. $\int \cos 2x \sin 3x dx$
33. $\int \frac{x^4+x^2-1}{x^3+x}dx$
34. $\int \frac{x^3dx}{\sqrt{x^2+4}}$
35. $\int \frac{dx}{(x^2-1)^2}$
36. $\int \frac{dx}{1+\sqrt{x}}$
37. $\int \cos\sqrt{x}dx$
38. $\int \tan^4 x dx$
39. $\int \frac{dx}{\sqrt{x}(x-1)}$
40. $\int x^2\tan^{-1} x dx$
41. $\int \sin^{-1} x dx$
42. $\int \frac{\sqrt{x}dx}{\sqrt{1-x}}$
43. $\int \frac{\sqrt{x+1}}{x}dx$
44. $\int \sqrt{x}\sqrt{1-x}dx$

Section 1.1: Substitution

1. $-\frac{2}{5}\sqrt{2-5x}+C$
2. $\frac{1}{2}e^{2x}-e^{x}+x+C$
3. $-\sqrt{1-x^2}+C$
4. $\frac{1}{4}(1+x^3)^{\frac{4}{3}}+C$
5. $-\frac{1}{2(1+x^2)}+C$
6. $2\tan^{-1}\sqrt{x}+C$
7. $\cos\frac{1}{x}+C$
8. $-\frac{1}{2}e^{-x^2}+C$
9. $\frac{1}{3}(\ln x)^3+C$
10. $\ln(2+e^{x})+C$
11. $\tan^{-1}e^{x}+C$
12. $2\sin\sqrt{x}+C$
13. $-\ln|\cos x|+C$
14. $x-\ln(1+e^{x})+C$
15. $\frac{1}{200}(x^2+2)^{100}+C$
16. $-\sqrt{25-x^2}+C$
17. $\frac{1}{3}\sqrt{3x^2+1}+C$
18. $-\frac{2}{3}\sqrt{9-x^3}+C$
19. $\frac{(x+2)^{101}}{101}-\frac{(x+2)^{100}}{50}+C.$
20. $\frac{1}{12}(2x-5)\sqrt{4x+5}+C$
21. $\frac{2}{15}(x-1)^{3/2}(3x+2)+C$
22. $\frac{2}{5}(x-1)^{3/2}(x+4)+C$
23. $\frac{2}{3}(x-18)\sqrt{x+9}+C$
24. $\frac{1}{135}\left(3x^2+1\right)^{3/2}\left(9x^2-2\right)+C$

Section 1.2: Integration By Parts

1. $x\ln x-x+C$
2. $\frac{x^3}{3}(\ln x-\frac{1}{3})+C$
3. $-\frac{1}{x}((\ln x)^2+2\ln x+2)+C$
4. $-(x+1)e^{-x}+C$
5. $-\frac{e^{-2x}}{4}(2x^2+2x+1)+C$
6. $x\sin x+\cos x+C$
7. $-\frac{2x^2-1}{4}\cos 2x+\frac{x}{2}\sin 2x+C$
8. $x(\ln x)^2-2x\ln x+2x+C$
9. $x\sin^{-1}x+\sqrt{1-x^2}+C$
10. $-\frac{x}{2}+\frac{1+x^2}{2}\tan^{-1}x+C$
11. $x\ln(x+\sqrt{1+x^2})-\sqrt{1+x^2}+C$
12. $\frac{x^2}{4}-\frac{x}{4}\sin 2x-\frac{1}{8}\cos 2x+C$
13. $\frac{x}{2}(\sin(\ln x)-\cos(\ln x))+C$
14. $\frac{1}{16}\sin 4x-\frac{1}{4}x\cos 4x+C$
15. $\frac{x^2\cos^{-1}x}{2}+\frac{\sin^{-1}x}{4}-\frac{x\sqrt{1-x^2}}{4}+C$
16. $x\tan^{-1}x-\frac{1}{2}\log(x^2+1)+C$
17. $\frac{1}{100}x^{100}\ln x-\frac{x^{100}}{10000}+C$
18. $-\frac{1}{10000x^{100}}-\frac{\ln x}{100x^{100}}+C$
19. $x\tan x+\ln(\cos x)+C$
20. $\frac{1}{13}e^{2x}(3\sin 3x+2\cos 3x)+C.$

Section 1.4: Trigonometric Integrals

1. $-\cot\frac{x}{2}+C$
2. $\frac{1}{6}\sin^6 x+C$
3. $\frac{1}{4}\sin 2x-\frac{1}{16}\sin 8x+C$
4. $3\sin\frac{x}{6}+\frac{3}{5}\sin\frac{5x}{6}+C$
5. $\sin x-\frac{1}{3}\sin^3 x+C$
6. $\frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C$
7. $\frac{1}{3}\tan^3 x+C$
8. $\frac{1}{3}\sec^3 x+-\sec x+C$
9. $-x-\cot x+C$
10. $-\frac{1}{\sin x}+\frac{1}{2}\ln\frac{1+\sin x}{1-\sin x}+C$
11. $-\frac{1}{2}\cos^2 x+\frac{1}{2}\ln(1+\cos^2 x)+C$
12. $\frac{\tan^4}{4}-\frac{\tan^2 x}{2}-\ln|\cos x|+C$
13. $-8\cot 2x-\frac{8}{3}\cot^3 2x+C$
14. $-\frac{1}{8}\cos 4x-\frac{1}{12}\cos 6x+C$
15. $\frac{x}{4}+\frac{\sin 2x}{8}+\frac{\sin 4x}{16}+\frac{\sin 6x}{24}+C$
16. $\frac{\cos^8(x)}{8}-\frac{\cos^6(x)}{6}+C$
17. $\frac{\sin^9(x)}{9}-\frac{2\sin^7(x)}{7}+\frac{\sin^5(x)}{5}+C$
18. $-\frac{1}{6}\cos^5 x\sin x+\frac{1}{24}\cos^3 x\sin x+\frac{1}{16}\cos x\sin x+\frac{1}{16}x+C.$

Section 1.5: Trigonometric Substitution

1. $x-\tan^{-1}x+C$
2. $\frac{x}{\sqrt{1-x^2}}+C$
3. $-\sqrt{1-x^2}+\sin^{-1}x+C$
4. $\frac{x}{\sqrt{1+x^2}}+C$
5. $\frac{9}{2}\sin^{-1}\frac{x}{3}-\frac{x}{2}\sqrt{9-x^2}+C$
6. $\ln|x+\sqrt{4+x^2}|+C$
7. $\sqrt{16-x^2}\left(\frac{x^3}{4}-2x\right)+32\sin^{-1}\left(\frac{x}{4}\right)+C$
8. $-\frac{\sqrt{x^2+4}}{4x}+C$
9. $\frac{x}{\sqrt{4x^2+1}}$
10. $\frac{x-1}{\sqrt{2x-x^2}}$

Section 1.6: Rational Functions

1. $-x+\frac{1}{2}\ln|\frac{1+x}{1-x}|+C$
2. $9x-\frac{3}{2}x^2+\frac{1}{3}x^3-27\ln|3+x|+C$
3. $x+\ln(1+x^2)+C$
4. $\frac{1}{4}\ln|\frac{x-1}{x+3}|+C$
5. $\frac{1}{10\sqrt{2}}\ln|\frac{x-\sqrt{2}}{x+\sqrt{2}}|-\frac{1}{5\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}+C$
6. $\frac{1}{x+1}+\frac{1}{2}\ln|x^2-1|+C$
7. $-\frac{5x-6}{x^2-3x+2}+4\ln|\frac{x-1}{x-2}|+C$
8. $\tan^{-1}x+\frac{5}{6}\ln\frac{x^2+1}{x^2+4}+C$
9. $\frac{1}{2}\tan^{-1}x+\frac{1}{4}\ln\frac{(x+1)^2}{x^2+1}+C$
10. $x^2+2\ln|x+1|+3\ln|x-3|+C$
11. $\tan^{-1}x-\frac{1}{x-1}+\ln\frac{x^2+1}{(x-1)^2}+C$
12. $\frac{1}{2(x^2+1)}+\ln|x|-\frac{1}{2}\ln(x^2+1)+C$
13. $\frac{9}{2}\ln(x-3)-4\ln(x-2)+\frac{1}{2}\ln(x-1)+C$
14. $\frac{1}{4}\ln\left(\frac{x^2}{x^2-2x+2}\right)-\frac{1}{2}\tan^{-1}(1-x)+C$

Section 1.7: t-method

1. $-\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\ln|\tan\frac{x}{2}| + C$
2. $\tan x - \sec x + C$
3. $\frac{1}{\cos x} + \frac{1}{3\cos^3 x} + \ln|\tan\frac{x}{2}| + C$
4. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\left(\frac{x}{2}\right)+1}{\sqrt{3}}\right) + C$
5. $2\sqrt{2}\tan^{-1}\left(\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{2}}\right) - x + C$
6. $\frac{1}{2}(x + \ln(\sin x + \cos x + 3)) - \frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{2\tan\left(\frac{x}{2}\right)+1}{\sqrt{7}}\right) + C$

Section 1.8: Miscellaneous

1. $\frac{1}{3}(\ln x)^3 + C$
2. $\frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C$
3. $-\sqrt{1-x^2} + C$
4. $\ln|x+1| - \frac{3}{x+1} + C$
5. $-\frac{1}{\sin x} - \sin x + C$
6. $-\frac{1}{2(1+x^2)} + C$
7. $e^x - \ln(1+e^x) + C$
8. $\frac{1}{2}\ln|1 + 2\ln x| + C$
9. $\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$
10. $-\ln(1+\cos^2 x) + C$
11. $-e^{\frac{1}{x}} + C$
12. $\sec x + C$
13. $-\frac{x^2}{2} + x\tan x + \ln\cos x + C$
14. $-\ln|1 + \csc x| + C$
15. $\frac{1}{2}x^2 + \frac{1}{2}\ln|x^2 - 1| + C$
16. $-\frac{x}{2} + \frac{1}{3}\ln|e^x - 1| + C$
17. $-\frac{4}{3}\sqrt{1+\ln x} + \frac{2}{3}(\ln x)\sqrt{1+\ln x} + C$
18. $-\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\frac{x}{3} + C$
19. $x - \tan^{-1} x + C$
20. $\ln|x + \sqrt{x^2+9}| + C$
21. $\ln|\sin x| - \frac{1}{2}\sin^2 x + C$
22. $x + 17\ln|x-3| - 12\ln|x-2| + C$
23. $\frac{2}{3}(x-2)^{\frac{3}{2}} + 4(x-2)^{\frac{1}{2}} + C$
24. $x - 2\ln(1 + \sqrt{1+e^x}) + C$
25. $\frac{x}{2}(\cos(\ln x) + \sin(\ln x)) + C$
26. $\frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$
27. $-2\sin^{-1} e^{-\frac{x}{2}} + C$
28. $-\frac{\sqrt{4-x^2}}{x} + C$
29. $\frac{1}{x} - 2\ln|x| + 2\ln|x-1| + C$
30. $\frac{1}{3}\sec^3 x + C$
31. $\frac{1}{3}x^2(x^2+1)^{\frac{3}{2}} - \frac{2}{15}(x^2+1)^{\frac{5}{2}} + C$
32. $-\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + C$
33. $\frac{1}{2}x^2 - \ln|x| + \frac{1}{2}\ln(x^2+1) + C$
34. $\frac{1}{3}(x^2+4)^{\frac{3}{2}} - 4\sqrt{x^2} + C$
35. $\frac{1}{4}\ln|x+1| - \frac{1}{4}\ln|x-1| - \frac{x}{2(x^2-1)} + C$
36. $2\sqrt{x} - 2\ln(1+\sqrt{x}) + C$
37. $2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$
38. $\frac{1}{3}\tan^3 x - \tan x + x + C$
39. $\ln|\sqrt{x}-1| - \ln|\sqrt{x}+1| + C$
40. $\frac{1}{3}x^3\tan^{-1} x - \frac{1}{6}x^2 + \frac{1}{6}\ln(x^2+1) + C$
41. $x\sin^{-1} x + \sqrt{1-x^2} + C$
42. $\sin^{-1}\sqrt{x} - \sqrt{x}\sqrt{1-x} + C$
43. $2\sqrt{x+1} + \ln|\sqrt{x}-1| - \ln|\sqrt{x}+1| + C$
44. $\frac{1}{4}\sin^{-1}\sqrt{x} - \frac{1}{4}\sqrt{x}\sqrt{1-x}(1-2x) + C$