THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 1

Trigonometric functions In the following, the arguments of trigonometric functions are in radian ($1^\circ = \frac{\pi}{180}$ rad) and the symbol "rad" will be omitted.

The trigonometric functions are defined by

sine	$\sin x$	$=$	$\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$	
cosine	$\cos x$	$=$	$\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$	
tangent	$\tan x$	$=$	$\dfrac{\sin x}{\cos x}$	(when $\cos x \neq 0$)
cosecant	$\csc x$	$=$	$\dfrac{1}{\sin x}$	(when $\sin x \neq 0$)
secant	$\sec x$	$=$	$\dfrac{1}{\cos x}$	(when $\cos x \neq 0$)
cotangent	$\cot x$	$=$	$\dfrac{\cos x}{\sin x}$	(when $\sin x \neq 0$)

Remarks: There are many definitions for sine and cosine functions, such as unit circle, differential equation, e.t.c..

Trigonometric identities The trigonometric functions have the following identities

$$\sin^2 x + \cos^2 x = 1, \qquad \tan^2 x + 1 = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x$$

(Sum and difference formulas)

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\sin(2x) = 2 \sin x \cos x$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$	$\tan(2x) = \dfrac{2 \tan x}{1 - \tan^2 x}$

(Sum to product and product to sum formulas)

$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$	$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$
$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$	$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$
$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$	$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$

To prove these formula, one just need to show $\sin(x+y) = \sin x \cos y + \cos x \sin y$ and $\cos(x+y) = \cos x \cos y - \sin x \sin y$ first and others are just algebraic manipulation of these two.

Mathematical induction: To prove a collection of proposition $P(n)$ concerning natural numbers, we may use mathematical induction. In using mathematical induction, we have to prove the base case $P(1)$ is true and the induction hypothesis: Given a positive integer n, if $P(n)$ is true, then $P(n+1)$ is true. Then by property of natural numbers, the collection of natural numbers that $P(n)$ is true will be equal to that of natural numbers, that is, for all natural numbers n, $P(n)$ is true.

Remarks: (1) There are many variants of mathematical induction but the principles are the same.
(2) Some people think natural numbers are $0, 1, 2, 3, \ldots$ and some people think natural numbers are $1, 2, 3, \ldots$. Mathematical induction works on both of them, but the base case need to be changed to 0 in the first scenario.

Problems that may be demonstrated in class :

Q1. Show that for all real number x not equal to $\frac{n\pi}{2}$ for any integer n, we have

$$(\sin x + \cos x)(\tan x + \cot x) = (\sec x + \csc x).$$

Q2. Show that for all real number x not equal to $\frac{n\pi}{2}$ for any integer n, we have

$$\frac{\cos x}{1 \pm \sin x} = \frac{1 \mp \sin x}{\cos x}.$$

Q3. Show that for all real number x not equal to $\frac{n\pi}{2}$ for any integer n, we have

$$\sin x - \csc x = -\cot x \cos x.$$

Q4. Show that for all real number α, β, γ with $\alpha + \beta + \gamma = \pi$, we have

$$\sin \alpha + \sin \beta + \sin \gamma = 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}.$$

Q5. Prove that for all positive integer n, we have

$$\sum_{k=1}^{n} k^2 = 1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Q6. Prove that for all positive integer n, we have

$$\sum_{k=1}^{n} \sin k = \frac{\sin \frac{n+1}{2}}{\sin \frac{1}{2}} \sin \frac{n}{2}.$$

Q7. Prove that for all positive integer n and real number x not equal to a multiple of π, we have

$$\prod_{k=0}^{n-1} \cos(2^k x) = \cos x \cos(2x) \cos(4x) \cdots \cos(2^{n-1} x) = \frac{\sin(2^n x)}{2^n \sin x}.$$

Q8. Prove that for all positive integer n we have

$$53^n - 46^n - 31^n + 24^n \text{ is divisible by } 77.$$