Definition (Differentiability)

Let $f(x)$ be a function. Denote

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

and we say that $f(x)$ is **differentiable** at $x = a$ if the above limit exists. We say that $f(x)$ is differentiable on (a, b) if $f(x)$ is differentiable at every point in (a, b).

The above limit can also be written as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Theorem

If $f(x)$ differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

$$\textbf{Differentiable at } x = a \Rightarrow \textbf{Continuous at } x = a$$

Proof.

Suppose $f(x)$ is differentiable at $x = a$. Then

$$\begin{aligned}
\lim_{x \to a}(f(x) - f(a)) &= \lim_{x \to a}\left(\frac{f(x) - f(a)}{x - a}\right)(x - a) \\
&= \lim_{x \to a}\left(\frac{f(x) - f(a)}{x - a}\right)\lim_{x \to a}(x - a) \\
&= f'(a) \cdot 0 = 0
\end{aligned}$$

Therefore $f(x)$ is continuous at $x = a$. □

Note that the converse of the above theorem does not hold. The function $f(x) = |x|$ is continuous but not differentiable at 0.

Example

1. $f(x) = e^x$: $f'(0) = \lim_{h\to 0} \frac{e^h - e^0}{h} = \lim_{h\to 0} \frac{e^h - 1}{h} = 1.$
2. $f(x) = \ln x$: $f'(1) = \lim_{h\to 0} \frac{\ln(1+h) - \ln 1}{h} = \lim_{h\to 0} \frac{\ln(1+h)}{h} = 1.$
3. $f(x) = \sin x$: $f'(0) = \lim_{h\to 0} \frac{\sin h - \sin 0}{h} = \lim_{h\to 0} \frac{\sin h}{h} = 1.$

Example

Find the values of a, b if $f(x) = \begin{cases} 4x - 1, & \text{if } x \leq 1 \\ ax^2 + bx, & \text{if } x > 1 \end{cases}$ is differentiable at $x = 1$.

Solution

Since $f(x)$ is differentiable at $x = 1$, $f(x)$ is continuous at $x = 1$ and we have

$$\lim_{x\to 1^+} f(x) = f(1) \Rightarrow \lim_{x\to 1^+} (ax^2 + bx) = a + b = 3.$$

Moreover, $f(x)$ is differentiable at $x = 1$ and we have

$$\begin{aligned} \lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} &= \lim_{h\to 0^-} \frac{(4(1+h) - 1) - 3}{h} = 4 \\ \lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} &= \lim_{h\to 0^+} \frac{a(1+h)^2 - b(1+h) - 3}{h} = 2a + b \end{aligned}$$

Therefore $\begin{cases} a + b = 3 \\ 2a + b = 4 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 2 \end{cases}$.

Definition (First derivative)

Let $y = f(x)$ be a differentiable function on (a, b). The **first derivative** of $f(x)$ is the function on (a, b) defined by

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Theorem

Let $f(x)$ and $g(x)$ be differentiable functions and c be a real number. Then

1. $(f+g)'(x) = f'(x) + g'(x)$
2. $(cf)'(x) = cf'(x)$

Theorem

1. $\dfrac{d}{dx}x^n = nx^{n-1},\ n \in \mathbb{Z}^+,\ for\ x \in \mathbb{R}$
2. $\dfrac{d}{dx}e^x = e^x\ \ for\ x \in \mathbb{R}$
3. $\dfrac{d}{dx}\ln x = \dfrac{1}{x}\ \ for\ x > 0$
4. $\dfrac{d}{dx}\cos x = -\sin x\ \ for\ x \in \mathbb{R}$
5. $\dfrac{d}{dx}\sin x = \cos x\ \ for\ x \in \mathbb{R}$

Proof ($\dfrac{d}{dx}x^n = nx^{n-1}$)

Let $y = x^n$. For any $x \in \mathbb{R}$, we have

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}\\ &= \lim_{h\to 0}\frac{(x+h-x)((x+h)^{n-1}+(x+h)^{n-2}x+\cdots+x^{n-1})}{h}\\ &= \lim_{h\to 0}((x+h)^{n-1}+(x+h)^{n-2}x+\cdots+x^{n-1})\\ &= nx^{n-1}\end{aligned}$$

Note that the above proof is valid only when $n \in \mathbb{Z}^+$ is a positive integer.

Proof ($\frac{d}{dx}e^x = e^x$)

Let $y = e^x$. For any $x \in \mathbb{R}$, we have

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} = e^x.$$

(Alternative proof)

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) \\ &= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ &= e^x \end{aligned}$$

In general, differentiation cannot be applied term by term to infinite series. The second proof is valid only after we prove that this can be done to **power series**.

Proof

$\left(\frac{d}{dx}\ln x = \frac{1}{x}\right)$ *Let* $f(x) = \ln x$. *For any* $x > 0$, *we have*

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} = \lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{h} = \frac{1}{x}.$$

$\left(\frac{d}{dx}\cos x = -\sin x\right)$ *Let* $f(x) = \cos x$. *For any* $x \in \mathbb{R}$, *we have*

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0}\frac{-2\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} = -\sin x.$$

$\left(\frac{d}{dx}\sin x = \cos x\right)$ *Let* $f(x) = \sin x$. *For any* $x \in \mathbb{R}$, *we have*

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} = \cos x.$$

Definition

Let $a > 0$ be a positive real number. For $x \in \mathbb{R}$, we define

$$a^x = e^{x \ln a}.$$

Theorem

Let $a > 0$ be a positive real number. We have

1. $a^{x+y} = a^x a^y$ *for any* $x, y \in \mathbb{R}$
2. $\dfrac{d}{dx} a^x = a^x \ln a.$

Proof.

1. $a^{x+y} = e^{(x+y)\ln a} = e^{x \ln a} e^{y \ln a} = a^x a^y$
2. $\dfrac{d}{dx} a^x = \dfrac{d}{dx} e^{x \ln a} = e^{x \ln a} \ln a = a^x \ln a$

□

Example

Let $f(x) = |x|$ for $x \in \mathbb{R}$. Show that $f(x)$ is not differentiable at $x = 0$.

Proof.

Observe that

$$\begin{aligned}\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} &= \lim_{h\to 0^-} \frac{-h}{h} = -1\\ \lim_{h\to 0^+} \frac{f(h)-f(0)}{h} &= \lim_{h\to 0^+} \frac{h}{h} = 1\end{aligned}$$

Thus the limit

$$\lim_{h\to 0} \frac{f(h)-f(0)}{h}$$

does not exist. Therefore $f(x)$ is not differentiable at $x = 0$. □

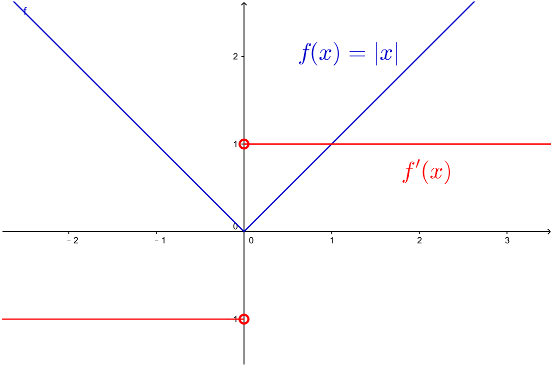

Figure: $f(x) = |x|$

Example

Let $f(x) = |x|x$ for $x \in \mathbb{R}$. Find $f'(x)$.

Solution

When $x < 0$, $f(x) = -x^2$ *and* $f'(x) = -2x$. *When* $x > 0$, $f(x) = x^2$ *and* $f'(x) = 2x$. *When* $x = 0$, *we have*

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(h) - f(0)}{h} &= \lim_{h\to 0^-} \frac{-h^2 - 0}{h} = 0 \\
\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} &= \lim_{h\to 0^+} \frac{h^2 - 0}{h} = 0
\end{aligned}$$

Thus $f'(0) = 0$. *Therefore*

$$\begin{aligned}
f'(x) &= \begin{cases} -2x, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 2x, & \text{if } x > 0 \end{cases} \\
&= 2|x|.
\end{aligned}$$

Note that $f'(x) = 2|x|$ is continuous at $x = 0$.

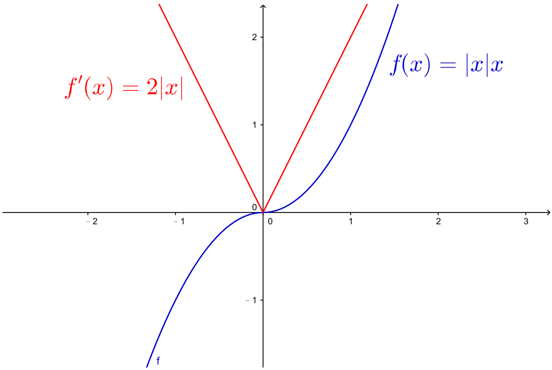

Figure: $f(x) = |x|x$

Example

Let

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

1. Find $f'(x)$ for $x \neq 0$.
2. Determine whether $f(x)$ is differentiable at $x = 0$.

Solution

1. When $x \neq 0$,

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}.$$

2. We have

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin \frac{1}{h}}{h} = \lim_{h \to 0} \sin \frac{1}{h}$$

does not exist. Therefore $f(x)$ *is not differentiable at* $x = 0$.

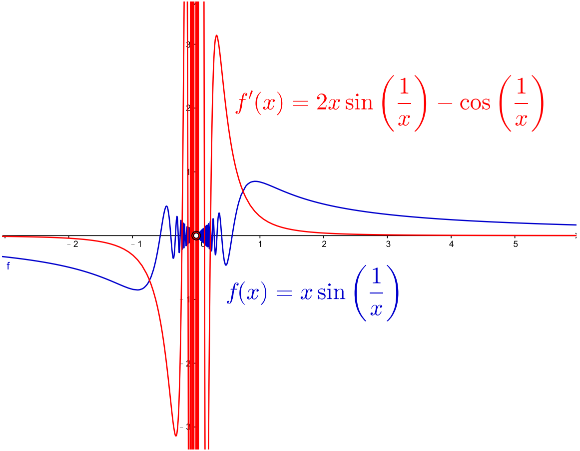

Figure: $f(x) = x\sin\left(\frac{1}{x}\right)$

Example

Let

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

1. Find $f'(x)$.
2. Determine whether $f'(x)$ is continuous at $x = 0$.

Solution

1. When $x \neq 0$, we have

$$f'(x) = 2x \sin \frac{1}{x} + x^2 \left(-\frac{1}{x^2} \cos \frac{1}{x} \right) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

Solution

2. *When $x = 0$, we have*

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h\to 0} h \sin \frac{1}{h}.$$

Since $\lim_{h\to 0} h = 0$ *and* $|\sin \frac{1}{h}| \leq 1$ *is bounded, we have* $f'(0) = 0$. *Therefore*

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

Observe that

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x} \right)$$

does not exist. We conclude that $f'(x)$ *is not continuous at* $x = 0$.

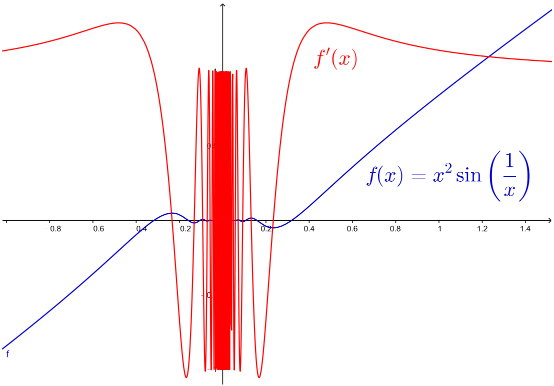

Figure: $f(x) = x^2 \sin\left(\frac{1}{x}\right)$

Example

$f(x)$	$f(x)$ is continuous at $x = 0$	$f(x)$ is differentiable at $x = 0$	$f'(x)$ is continuous at $x = 0$
$\lvert x \rvert$	Yes	No	Not applicable
$\lvert x \rvert x$	Yes	Yes	Yes
$x \sin\left(\frac{1}{x}\right)$; $f(0) = 0$	Yes	No	Not applicable
$x^2 \sin\left(\frac{1}{x}\right)$; $f(0) = 0$	Yes	Yes	No

Example

The following diagram shows the relations between the existence of limit, continuity and differentiability of a function at a point a. (Examples in the bracket is for $a = 0$.)

Second differentiable	$\left(f(x) = \frac{\sin x}{x};\ f(0) = 1\right)$
$\Downarrow$	
Continuously differentiable	$(f(x) = \lvert x \rvert x)$
$\Downarrow$	
Differentiable	$(f(x) = x^2 \sin(x^{-1});\ f(0) = 0)$
$\Downarrow$	
Continuous	$(f(x) = \lvert x \rvert)$
$\Downarrow$	
Limit exists	$\left(f(x) = \frac{\sin x}{x};\ f(0) = 0\right)$
$\Downarrow$	
Limit exists from both sides	$\left(f(x) = \frac{x}{\lvert x \rvert};\ f(0) = 0\right)$

Theorem (Basic formulas for differentiation)

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d}{dx}\cosh x = \sinh x \qquad \frac{d}{dx}\sinh x = \cosh x$$

Theorem (Product rule and quotient rule)

Let u and v be differentiable functions of x. Then

$$\begin{aligned}\frac{d}{dx}uv &= u\frac{dv}{dx}+v\frac{du}{dx}\\ \frac{d}{dx}\frac{u}{v} &= \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}\end{aligned}$$

Proof

Let $u=f(x)$ *and* $v=g(x)$.

$$\begin{aligned}\frac{d}{dx}uv &= \lim_{h\to0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to0}\left(\frac{f(x+h)g(x+h)-f(x+h)g(x)}{h}+\frac{f(x+h)g(x)-f(x)g(x)}{h}\right)\\ &= \lim_{h\to0}\left(f(x+h)\cdot\frac{g(x+h)-g(x)}{h}+g(x)\cdot\frac{f(x+h)-f(x)}{h}\right)\\ &= u\frac{dv}{dx}+v\frac{du}{dx}\end{aligned}$$

Proof.

$$\begin{aligned}
\frac{d}{dx}\frac{u}{v} &= \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{hg(x)g(x+h)} \\
&= \lim_{h\to 0}\left(\frac{f(x+h)g(x)-f(x)g(x)}{hg(x)g(x+h)}-\frac{f(x)g(x+h)-f(x)g(x)}{hg(x)g(x+h)}\right) \\
&= \lim_{h\to 0}\left(g(x)\cdot\frac{f(x+h)-f(x)}{hg(x)g(x+h)}-f(x)\cdot\frac{g(x+h)-g(x)}{hg(x)g(x+h)}\right) \\
&= \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}
\end{aligned}$$

□

Theorem (Chain rule)

Let $y = f(u)$ be a function of u and $u = g(x)$ be a function of x. Suppose $g(x)$ is differentiable at $x = a$ and $f(u)$ is differentiation at $u = g(a)$. Then $f \circ g(x) = f(g(x))$ is differentiable at $x = a$ and

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

In other words,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Proof

$$\begin{aligned}(f\circ g)'(a) &= \lim_{h\to 0}\frac{f(g(a+h))-f(g(a))}{h}\\ &= \lim_{h\to 0}\frac{f(g(a+h))-f(g(a))}{g(a+h)-g(a)}\lim_{h\to 0}\frac{g(a+h)-g(a)}{h}\\ &= \lim_{k\to 0}\frac{f(g(a)+k)-f(g(a))}{k}\lim_{h\to 0}\frac{g(a+h)-g(a)}{h}\\ &= f'(g(a))g'(a)\end{aligned}$$

The above proof is valid only if $g(a+h)-g(a)\neq 0$ whenever h is sufficiently close to 0. This is true when $g'(a)\neq 0$ because of the following proposition.

Proposition

Suppose $g(x)$ is a function such that $g'(a)\neq 0$. Then there exists $\delta>0$ such that if $0<|h|<\delta$, then

$$g(a+h)-g(a)\neq 0.$$

When $g'(a) = 0$, we need another proposition.

Proposition

Suppose $f(u)$ is a function which is differentiable at $u = b$. Then there exists $\delta > 0$ and $M > 0$ such that

$$|f(b+h) - f(b)| < M|h| \text{ for any } |h| < \delta.$$

The proof of chain rule when $g'(a) = 0$ goes as follows. There exists $\delta > 0$ such that

$$|f(g(a+h)) - f(g(a))| < M|g(a+h) - g(a)| \text{ for any } |h| < \delta.$$

Therefore

$$\lim_{h \to 0} \left| \frac{f(g(a+h)) - f(g(a))}{h} \right| \leq \lim_{h \to 0} M \left| \frac{g(a+h) - g(a)}{h} \right| = 0$$

which implies $(f \circ g)'(a) = 0$.

Example

The chain rule is used in the following way. Suppose u is a differentiable function of x. Then

$$\begin{aligned}
\frac{d}{dx}u^n &= nu^{n-1}\frac{du}{dx} \\
\frac{d}{dx}e^u &= e^u\frac{du}{dx} \\
\frac{d}{dx}\ln u &= \frac{1}{u}\frac{du}{dx} \\
\frac{d}{dx}\cos u &= -\sin u\frac{du}{dx} \\
\frac{d}{dx}\sin u &= \cos u\frac{du}{dx}
\end{aligned}$$

Example

1. $$\frac{d}{dx}\sin^3 x = 3\sin^2 x\frac{d}{dx}\sin x = 3\sin^2 x\cos x$$

2. $$\frac{d}{dx}e^{\sqrt{x}} = e^{\sqrt{x}}\frac{d}{dx}\sqrt{x} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

3. $$\frac{d}{dx}\frac{1}{(\ln x)^2} = -\frac{2}{(\ln x)^3}\frac{d}{dx}\ln x = -\frac{2}{x(\ln x)^3}$$

4. $$\frac{d}{dx}\ln\cos 2x = \frac{1}{\cos 2x}(-\sin 2x)\cdot 2 = -\frac{2\sin 2x}{\cos 2x} = -2\tan 2x$$

5. $$\frac{d}{dx}\tan\sqrt{1+x^2} = \sec^2\sqrt{1+x^2}\cdot\frac{1}{2\sqrt{1+x^2}}\cdot 2x = \frac{x\sec^2\sqrt{1+x^2}}{\sqrt{1+x^2}}$$

6. $$\frac{d}{dx}\sec^3\sqrt{\sin x} = 3\sec^2\sqrt{\sin x}(\sec\sqrt{\sin x}\tan\sqrt{\sin x})\frac{1}{2\sqrt{\sin x}}\cdot\cos x$$
$$= \frac{3\sec^3\sqrt{\sin x}\tan\sqrt{\sin x}\cos x}{2\sqrt{\sin x}}$$

Example

7. $$\frac{d}{dx}\cos^4 x \sin x = \cos^4 x \cos x + 4\cos^3 x(-\sin x)\sin x = \cos^5 x - 4\cos^3 x \sin^2 x$$

8. $$\frac{d}{dx}\frac{\sec 2x}{\ln x} = \frac{\ln x(2\sec 2x \tan 2x) - \sec 2x(\frac{1}{x})}{(\ln x)^2} = \frac{\sec 2x(2x\tan 2x \ln x - 1)}{x(\ln x)^2}$$

9. $$e^{\frac{\tan x}{x}} = e^{\frac{\tan x}{x}}\left(\frac{x\sec^2 x - \tan x}{x^2}\right)$$

10. $$\sin\left(\frac{\ln x}{\sqrt{1+x^2}}\right) = \cos\left(\frac{\ln x}{\sqrt{1+x^2}}\right)\left(\frac{\sqrt{1+x^2}(\frac{1}{x}) - \ln x(\frac{2x}{2\sqrt{1+x^2}})}{1+x^2}\right) = \left(\frac{1+x^2-x^2\ln x}{x(1+x^2)^{\frac{3}{2}}}\right)\cos\left(\frac{\ln x}{\sqrt{1+x^2}}\right)$$

Definition (Implicit functions)

An **implicit function** is an equation of the form $F(x,y)=0$. An implicit function may not define a function. Sometimes it defines a function when the domain and range are specified.

Theorem

Let $F(x,y)=0$ be an implicit function. Then

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0$$

and we have

$$\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}.$$

Here $\frac{\partial F}{\partial x}$ is called the partial derivative of F with respect to x which is the derivative of F with respect to x while considering y as constant. Similarly the partial derivative $\frac{\partial F}{\partial y}$ is the derivative of F with respect to y while considering x as constant.

Example

Find $\dfrac{dy}{dx}$ for the following implicit functions.

1. $x^2 - xy - xy^2 = 0$
2. $\cos(xe^y) + x^2 \tan y = 1$

Solution

1.
$$\begin{aligned} 2x - (y + xy') - (y^2 + 2xyy') &= 0 \\ xy' + 2xyy' &= 2x - y - y^2 \\ y' &= \frac{2x - y - y^2}{x + 2xy} \end{aligned}$$

2.
$$\begin{aligned} -\sin(xe^y)(e^y + xe^y y') + 2x \tan y + x^2 \sec^2 y y' &= 0 \\ x^2 \sec^2 y y' - xe^y \sin(xe^y) y' &= e^y \sin(xe^y) - 2x \tan y \\ y' &= \frac{e^y \sin(xe^y) - 2x \tan y}{x^2 \sec^2 y - xe^y \sin(xe^y)} \end{aligned}$$

Theorem

Suppose $f(y)$ is a differentiable function with $f'(y) \neq 0$ for any y. Then the inverse function $y = f^{-1}(x)$ of $f(y)$ is differentiable and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

In other words,

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Proof.

By chain rule, we have

$$\begin{aligned} f(f^{-1}(x)) &= x \\ f'(f^{-1}(x))(f^{-1})'(x) &= 1 \\ (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \end{aligned}$$

□

Theorem

1. *For* $\sin^{-1}: [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$,
$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}.$$
2. *For* $\cos^{-1}: [-1, 1] \to [0, \pi]$,
$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}.$$
3. *For* $\tan^{-1}: \mathbb{R} \to [-\frac{\pi}{2}, \frac{\pi}{2}]$,
$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}.$$

Proof.

1.

$$\begin{aligned} y &= \sin^{-1} x \\ \sin y &= x \\ \cos y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos y} \\ &= \frac{1}{\sqrt{1-\sin^2 y}} \quad (\text{Note: } \cos y \geq 0 \text{ for } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}) \\ &= \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

The other parts can be proved similarly. □

Example

Find $\frac{dy}{dx}$ if $y = x^x$.

Solution

There are 2 methods.

Method 1. Note that $y = x^x = e^{x \ln x}$. *Thus*

$$\frac{dy}{dx} = e^{x \ln x}(1 + \ln x) = x^x(1 + \ln x).$$

Method 2. Taking logarithm on both sides, we have

$$\begin{aligned} \ln y &= x \ln x \\ \frac{1}{y}\frac{dy}{dx} &= 1 + \ln x \\ \frac{dy}{dx} &= y(1 + \ln x) \\ &= x^x(1 + \ln x) \end{aligned}$$

Example

Let u and v be functions of x. Show that

$$\frac{d}{dx}u^v = u^v v' \ln u + u^{v-1} v u'.$$

Proof.

We have

$$\begin{aligned}
\frac{d}{dx}u^v &= \frac{d}{dx}e^{v\ln u} \\
&= e^{v\ln u}\left(\left(v'\ln u + v\cdot\frac{u'}{u}\right)\right. \\
&= u^v v'\left(\ln u + \frac{vu'}{u}\right) \\
&= u^v v'\ln u + u^{v-1}vu'
\end{aligned}$$

□

Definition (Second and higher derivatives)

Let $y = f(x)$ be a function. The **second derivative** of $f(x)$ is the function

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

The second derivative of $y = f(x)$ is also denoted as $f''(x)$ or y''. Let n be a non-negative integer. The ***n*-th derivative** of $y = f(x)$ is defined inductively by

$$\begin{aligned} \frac{d^ny}{dx^n} &= \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) \quad \text{for } n \geq 1 \\ \frac{d^0y}{dx^0} &= y \end{aligned}$$

The n-th derivative is also denoted as $f^{(n)}(x)$ or $y^{(n)}$. Note that $f^{(0)}(x) = f(x)$.

Example

Find $\dfrac{d^2y}{dx^2}$ for the following functions.

1. $y = \ln(\sec x + \tan x)$
2. $x^2 - y^2 = 1$

Solution

1.
$$\begin{aligned} y' &= \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x) \\ &= \sec x \\ y'' &= \sec x \tan x \end{aligned}$$

2.
$$\begin{aligned} 2x - 2yy' &= 0 \\ y' &= \frac{x}{y} \\ y'' &= \frac{y - xy'}{y^2} \\ &= \frac{y - x(\frac{x}{y})}{y^2} \\ &= \frac{y^2 - x^2}{y^3} \end{aligned}$$

Theorem (Leinbiz's rule)

Let u and v be differentiable function of x. Then

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)} v^{(k)}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ *is the binormial coefficient.*

Example

$$\begin{aligned}
(uv)^{(0)} &= u^{(0)}v^{(0)} \\
(uv)^{(1)} &= u^{(1)}v^{(0)} + u^{(0)}v^{(1)} \\
(uv)^{(2)} &= u^{(2)}v^{(0)} + 2u^{(1)}v^{(1)} + u^{(0)}v^{(2)} \\
(uv)^{(3)} &= u^{(3)}v^{(0)} + 3u^{(2)}v^{(1)} + 3u^{(1)}v^{(2)} + u^{(0)}v^{(3)} \\
(uv)^{(4)} &= u^{(4)}v^{(0)} + 4u^{(3)}v^{(1)} + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + u^{(0)}v^{(4)} \\
(uv)^{(5)} &= u^{(5)}v^{(0)} + 5u^{(4)}v^{(1)} + 10u^{(3)}v^{(2)} + 10u^{(2)}v^{(3)} + 5u^{(1)}v^{(4)} + u^{(0)}v^{(5)}
\end{aligned}$$

Proof

We prove the Leibniz's rule by induction on n. When $n = 0$, $(uv)^{(0)} = uv = u^{(0)}v^{(0)}$. *Assume that for some nonnegative m,*

$$(uv)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} u^{(m-k)}v^{(k)}.$$

Then

$$\begin{aligned} & (uv)^{(m+1)} \\ = \ & \frac{d}{dx}(uv)^{(m)} \\ = \ & \frac{d}{dx}\sum_{k=0}^{m} \binom{m}{k} u^{(m-k)}v^{(k)} \\ = \ & \sum_{k=0}^{m} \binom{m}{k} (u^{(m-k+1)}v^{(k)} + u^{(m-k)}v^{(k+1)}) \end{aligned}$$

Proof.

$$
\begin{aligned}
&= \sum_{k=0}^{m} \binom{m}{k} u^{(m-k+1)} v^{(k)} + \sum_{k=0}^{m} \binom{m}{k} u^{(m-k)} v^{(k+1)} \\
&= \sum_{k=0}^{m} \binom{m}{k} u^{(m-k+1)} v^{(k)} + \sum_{k=1}^{m+1} \binom{m}{k-1} u^{(m-(k-1))} v^{(k)} \\
&= \sum_{k=0}^{m} \binom{m}{k} u^{(m-k+1)} v^{(k)} + \sum_{k=1}^{m+1} \binom{m}{k-1} u^{(m-k+1)} v^{(k)} \\
&= \sum_{k=0}^{m+1} \left(\binom{m}{k} + \binom{m}{k-1} \right) u^{(m-k+1)} v^{(k)} \\
&= \sum_{k=0}^{m+1} \binom{m+1}{k} u^{(m+1-k)} v^{(k)}
\end{aligned}
$$

Here we use the convention $\binom{m}{-1} = \binom{m}{m+1} = 0$ in the second last equality. This completes the induction step and the proof of the Leibniz's rule. □

Example

Let $y = x^2 e^{3x}$. Find $y^{(n)}$ where n is a nonnegative integer.

Solution

Let $u = x^2$ *and* $v = e^{3x}$. *Then* $u^{(0)} = x^2$, $u^{(1)} = 2x$, $u^{(2)} = 2$ *and* $u^{(k)} = 0$ *for* $k \geq 3$. *On the other hand,* $v^{(k)} = 3^k e^{3x}$ *for any* $k \geq 0$. *Therefore by Leibniz's rule, we have*

$$\begin{aligned}
y^{(n)} &= \binom{n}{0} u^{(0)} v^{(n)} + \binom{n}{1} u^{(1)} v^{(n-1)} + \binom{n}{2} u^{(2)} v^{(n-2)} \\
&= x^2(3^n e^{3x}) + n(2x)(3^{n-1} e^{3x}) + \frac{n(n-1)}{2!}(2)(3^{n-2} e^{3x}) \\
&= (3^n x^2 + 2 \cdot 3^{n-1} nx + 3^{n-2}(n^2 - n))e^{3x} \\
&= 3^{n-2}(9x^2 + 6nx + n^2 - n)e^{3x}
\end{aligned}$$

Theorem

Let f be a function on (a, b) and $\xi \in (a, b)$ such that

1. *f is differentiable at $x = \xi$.*
2. *Either $f(x) \leq f(\xi)$ for any $x \in (a, b)$, or $f(x) \geq f(\xi)$ for any $x \in (a, b)$.*

Then $f'(\xi) = 0$.

Proof.

Suppose $f(x) \leq f(\xi)$ for any $x \in (a, b)$. The proof for the other case is more or less the same. For any $h < 0$ with $a < \xi + h < \xi$, we have $f(\xi + h) - f(\xi) \leq 0$ and h is negative. Thus

$$f'(\xi) = \lim_{h \to 0^-} \frac{f(\xi + h) - f(\xi)}{h} \geq 0$$

On the other hand, for any $h > 0$ with $\xi < \xi + h < b$, we have $f(\xi + h) - f(\xi) \leq 0$ and h is positive. Thus we have

$$f'(\xi) = \lim_{h \to 0^+} \frac{f(\xi + h) - f(\xi)}{h} \leq 0$$

Therefore $f'(\xi) = 0$. □

Theorem (Rolle's theorem)

Suppose $f(x)$ is a function which satisfies the following conditions.

1. $f(x)$ *is continuous on* $[a, b]$.
2. $f(x)$ *is differentiable on* (a, b).
3. $f(a) = f(b)$

Then there exists $\xi \in (a, b)$ *such that* $f'(\xi) = 0$.

Proof.

By extreme value theorem, there exist $a \leq \alpha, \beta \leq b$ such that

$$f(\alpha) \leq f(x) \leq f(\beta) \text{ for any } x \in [a, b].$$

Since $f(a) = f(b)$, at least one of α, β can be chosen in (a, b) and we let it be ξ. Then we have $f'(\xi) = 0$ since $f(x)$ attains its maximum or minimum at ξ. □

Theorem (Lagrange's mean value theorem)

Suppose $f(x)$ is a function which satisfies the following conditions.

1. $f(x)$ *is continuous on* $[a, b]$.
2. $f(x)$ *is differentiable on* (a, b).

Then there exists $\xi \in (a, b)$ *such that*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Proof

Let $g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$. *Since* $g(a) = g(b) = f(a)$, *by Rolle's theorem, there exists* $\xi \in (a, b)$ *such that*

$$\begin{aligned} g'(\xi) &= 0 \\ f'(\xi) - \frac{f(b) - f(a)}{b - a} &= 0 \\ f'(\xi) &= \frac{f(b) - f(a)}{b - a} \end{aligned}$$

Exercise (True or False)

Suppose $f(x)$ is a function which is differentiable on (a, b).

1. *$f(x)$ is constant on (a, b) if and only if $f'(x) = 0$ on (a, b).*
   Answer: **T**
2. *$f(x)$ is monotonic increasing on (a, b) if and only if $f'(x) \geq 0$ on (a, b).*
   Answer: **T**
3. *If $f(x)$ is strictly increasing on (a, b), then $f'(x) > 0$ on (a, b).*
   Answer: **F**
4. *If $f'(x) > 0$ on (a, b), then $f(x)$ is strictly increasing on (a, b).*
   Answer: **T**

$$f'(x) > 0$$
$$\Downarrow$$
Strictly increasing
$$\Downarrow$$
Monotonic increasing $\Leftrightarrow$ $f'(x) \geq 0$

Theorem

Let $f(x)$ be a function which is differentiable on (a, b). Then $f(x)$ is monotonic increasing if and only if $f'(x) \geq 0$ for any $x \in (a, b)$.

Proof

Suppose $f(x)$ is monotonic increasing on (a, b). Then for any $x \in (a, b)$, we have $f(x+h) - f(x) \geq 0$ for any $h > 0$ and thus

$$f'(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \geq 0.$$

On the other hand, suppose $f'(x) \geq 0$ for any $x \in (a, b)$. Then for any $\alpha < \beta$ in (a, b), by Lagrange's mean value theorem, there exists $\xi \in (\alpha, \beta)$ such that

$$f(\beta) - f(\alpha) = f'(\xi)(\beta - \alpha) \geq 0.$$

Therefore $f(x)$ is monotonic increasing on (a, b).

Corollary

$f(x)$ is constant on (a, b) if and only if $f'(x) = 0$ for any $x \in (a, b)$.

Theorem

If $f(x)$ is a differentiable function such that $f'(x) > 0$ for any $x \in (a, b)$, then $f(x)$ is strictly increasing.

Proof.

Suppose $f'(x) > 0$ for any $x \in (a, b)$. For any $\alpha < \beta$ in (a, b), by Lagrange's mean value theorem, there exists $\xi \in (\alpha, \beta)$ such that

$$f(\beta) - f(\alpha) = f'(\xi)(\beta - \alpha) > 0.$$

Therefore $f(x)$ is strictly increasing on (a, b). □

The converse of the above theorem is false.

Example

$f(x) = x^3$ is strictly increasing on $\mathbb{R}$ but $f'(0) = 0$ is not positive.

Theorem (Cauchy's mean value theorem)

Suppose $f(x)$ and $g(x)$ are functions which satisfies the following conditions.

1. $f(x), g(x)$ *is continuous on* $[a, b]$.
2. $f(x), g(x)$ *is differentiable on* (a, b).
3. $g'(x) \neq 0$ *for any* $x \in (a, b)$.

Then there exists $\xi \in (a, b)$ *such that*

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof

Let $h(x) = f(x) - \dfrac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a))$.

Since $h(a) = h(b) = f(a)$, by Rolle's theorem, there exists $\xi \in (a, b)$ such that

$$\begin{aligned} f'(\xi) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(\xi) &= 0 \\ \frac{f'(\xi)}{g'(\xi)} &= \frac{f(b) - f(a)}{g(b) - g(a)} \end{aligned}$$

Theorem (L'Hopital's rule)

Let $a \in [-\infty, +\infty]$. Suppose f and g are differentiable functions such that

1. $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ *(or $\pm\infty$).*
2. $g'(x) \neq 0$ *for any $x \neq a$ (on a neighborhood of a).*
3. $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$.

Then the limit of $\frac{f(x)}{g(x)}$ *at $x = a$ exists and* $\lim_{x \to a} \frac{f(x)}{g(x)} = L$.

Proof

For any $x \neq 0$, by Cauchy's mean value theorem, there exists ξ between a and x such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)}.$$

Here we redefine $f(a) = g(a) = 0$, if necessary, so that f and g are continuous at a. Note that $\xi \to a$ as $x \to a$. We have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(\xi)}{g'(\xi)} = L.$$

Example (Indeterminate form of types $\frac{0}{0}$ and $\frac{\infty}{\infty}$)

1. $\lim_{x\to 0} \frac{\sin x - x\cos x}{x^3} = \lim_{x\to 0} \frac{x\sin x}{3x^2} = \frac{1}{3}$

2. $\lim_{x\to 0} \frac{x^2}{\ln \sec x} = \lim_{x\to 0} \frac{2x}{\frac{\sec x \tan x}{\sec x}} = \lim_{x\to 0} \frac{2x}{\tan x} = \lim_{x\to 0} \frac{2}{\sec^2 x} = 2$

3. $\lim_{x\to 0} \frac{\ln(1+x^3)}{x - \sin x} = \lim_{x\to 0} \frac{\frac{x^2}{1+x^3}}{1-\cos x} = \lim_{x\to 0} \frac{1}{1+x^3} \lim_{x\to 0} \frac{x^2}{1-\cos x}$
$= \lim_{x\to 0} \frac{2x}{\sin x} = 2$

4. $\lim_{x\to +\infty} \frac{\ln(1+x^4)}{\ln(1+x^2)} = \lim_{x\to +\infty} \frac{\frac{4x^3}{1+x^4}}{\frac{2x}{1+x^2}} = \lim_{x\to +\infty} \frac{4x^3(1+x^2)}{2x(1+x^4)} = 2$

Example (Indeterminate form of types $\infty - \infty$ and $0 \cdot \infty$)

5. $$\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x} = \lim_{x\to 1}\frac{1-\frac{1}{x}}{\frac{x-1}{x}+\ln x} = \lim_{x\to 1}\frac{x-1}{x-1+x\ln x} = \lim_{x\to 1}\frac{1}{2+\ln x} = \frac{1}{2}$$

6. $$\lim_{x\to 0}\cot 3x\tan^{-1}x = \lim_{x\to 0}\frac{\tan^{-1}x}{\tan 3x} = \lim_{x\to 0}\frac{\frac{1}{1+x^2}}{3\sec^2 3x} = \lim_{x\to 0}\frac{1}{3(1+x^2)\sec^2 3x} = \frac{1}{3}$$

7. $$\lim_{x\to 0^+} x\ln\sin x = \lim_{x\to 0^+}\frac{\ln\sin x}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}\frac{-x^2\cos x}{\sin x} = 0$$

8. $$\lim_{x\to+\infty} x\ln\left(\frac{x+1}{x-1}\right) = \lim_{x\to+\infty}\frac{\ln(x+1)-\ln(x-1)}{\frac{1}{x}} = \lim_{x\to+\infty}\frac{\frac{1}{x+1}-\frac{1}{x-1}}{-\frac{1}{x^2}} = \lim_{x\to+\infty}\frac{2x^2}{(x+1)(x-1)} = 2$$

Example (Indeterminate form of types 0^0, 1^∞ and ∞^0)

Evaluate the following limits.

1. $\displaystyle\lim_{x\to 0^+} x^{\sin x}$
2. $\displaystyle\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$
3. $\displaystyle\lim_{x\to +\infty} (1+2x)^{\frac{1}{3\ln x}}$

Solution

1. $$\ln\left(\lim_{x\to0^+} x^{\sin x}\right) = \lim_{x\to0^+}\ln(x^{\sin x}) = \lim_{x\to0^+}\sin x\ln x = \lim_{x\to0^+}\frac{\ln x}{\csc x}$$
$$= \lim_{x\to0^+}\frac{\frac{1}{x}}{-\csc x\cot x} = \lim_{x\to0^+}\frac{-\sin^2 x}{x\cos x} = 0.$$
Thus $\lim_{x\to0^+} x^{\sin x} = e^0 = 1.$

2. $$\ln\left(\lim_{x\to0}(\cos x)^{\frac{1}{x^2}}\right) = \lim_{x\to0}\ln(\cos x)^{\frac{1}{x^2}} = \lim_{x\to0}\frac{\ln\cos x}{x^2} = \lim_{x\to0}\frac{-\tan x}{2x}$$
$$= \lim_{x\to0}\frac{-\sec^2 x}{2} = -\frac{1}{2}.$$
Thus $\lim_{x\to0}(\cos x)^{\frac{1}{x^2}} = e^{-\frac{1}{2}}.$

3. $$\ln\left(\lim_{x\to+\infty}(1+2x)^{\frac{3}{\ln x}}\right) = \lim_{x\to+\infty}\frac{3\ln(1+2x)}{\ln x} = \lim_{x\to+\infty}\frac{6x}{1+2x} = 3.$$
Thus $\lim_{x\to+\infty}(1+2x)^{\frac{1}{3\ln x}} = e^3.$

Example

The following shows some wrong use of L'Hopital rule.

1. $\lim_{x\to 0}\frac{\sec x - 1}{e^{2x}-1} = \lim_{x\to 0}\frac{\sec x\tan x}{2e^{2x}} = \lim_{x\to 0}\frac{\sec^2 x\tan x + \sec^3 x}{4e^{2x}} = \frac{1}{4}$
   This is wrong because $\lim_{x\to 0} e^{2x} \neq 0, \pm\infty$. One cannot apply L'Hopital rule $\lim_{x\to 0}\frac{\sec x\tan x}{2e^{2x}}$. The correct solution is
   $$\lim_{x\to 0}\frac{\sec x - 1}{e^{2x}-1} = \lim_{x\to 0}\frac{\sec x\tan x}{2e^{2x}} = 0.$$
2. $\lim_{x\to +\infty}\frac{5x - 2\cos^2 x}{3x+\sin^2 x} = \lim_{x\to +\infty}\frac{5+2\cos x\sin x}{3+\sin x\cos x} = \lim_{x\to +\infty}\frac{2(\cos^2 x - \sin^2 x)}{\cos^2 x - \sin^2 x} = 2$
   This is wrong because $\lim_{x\to +\infty}(5+2\cos x\sin x)$ and $\lim_{x\to +\infty}(3+\cos x\sin x)$ do not exist. One cannot apply L'Hopital rule to $\lim_{x\to +\infty}\frac{5+2\cos x\sin x}{3+\sin x\cos x}$. The correct solution is
   $$\lim_{x\to +\infty}\frac{5x - 2\cos^2 x}{3x+\sin^2 x} = \lim_{x\to +\infty}\frac{5 - \frac{2\cos^2 x}{x}}{3+\frac{\sin^2 x}{x}} = \frac{5}{3}.$$

Definition (Taylor polynomial)

Let $f(x)$ be a function such that the n-th derivative exists at $x = a$. The **Taylor polynomial** of degree n of $f(x)$ at $x = a$ is the polynomial

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Theorem

The Taylor polynomial $p_n(x)$ of degree n of $f(x)$ at $x = a$ is the unique polynomial such that

$$p_n^{(k)}(a) = f^{(k)}(a) \text{ for } k = 0, 1, 2, \ldots, n.$$

Example

Let $f(x)=\sqrt{1+x}=(1+x)^{\frac{1}{2}}$. The first four derivatives of $f(x)$ are

$$f'(x)=\frac{1}{2}(1+x)^{-\frac{1}{2}};\qquad f^{(3)}(x)=\frac{1\cdot 3}{2^3}(1+x)^{-\frac{5}{2}}$$
$$f''(x)=-\frac{1}{2^2}(1+x)^{-\frac{3}{2}};\quad f^{(4)}(x)=-\frac{1\cdot 3\cdot 5}{2^4}(1+x)^{-\frac{7}{2}}$$

The k-th derivative of $f(x)$ at $x=0$ is

$$f^{(k)}(0)=\frac{(-1)^{k+1}(2k-3)!!}{2^k}=\frac{(-1)^{k+1}\cdot 1\cdot 3\cdot 5\cdots(2k-5)(2k-3)}{2^k}.$$

Therefore the Taylor polynomial of $f(x)$ of degree n at $x=0$ is

$$\begin{aligned}p_n(x) &= f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\cdots+\frac{f^{(n)}(0)}{n!}x^n\\ &= 1+\frac{1}{2}x-\frac{1}{2!}\cdot\frac{1}{2^2}x^2+\frac{1}{3!}\cdot\frac{1\cdot 3}{2^3}x^3+\cdots+\frac{1}{n!}\cdot\frac{(2n-3)!!}{2^n}x^n\\ &= 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\cdots+\frac{(-1)^{n+1}(2n-3)!!x^n}{2^n n!}\end{aligned}$$

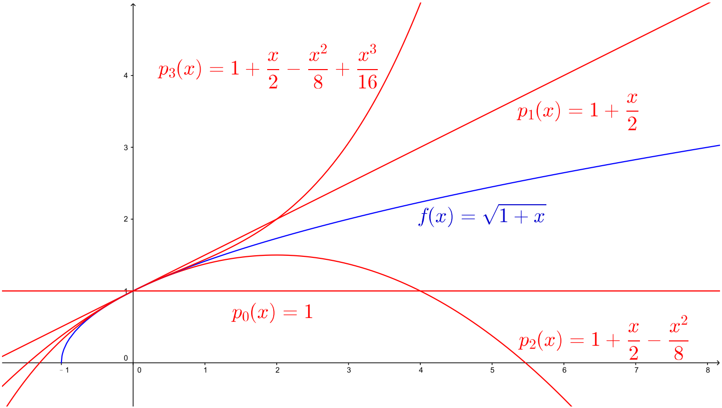

Figure: Taylor polynomials for $f(x) = \sqrt{1+x}$ at $x = 0$

Example

Let $f(x) = \cos x$. The n-th derivatives of $f(x)$ is

$$\frac{d^n}{dx^n}\cos x = \begin{cases} (-1)^k \cos x, & \text{if } n = 2k \text{ is even} \\ (-1)^k \sin x, & \text{if } n = 2k-1 \text{ is odd} \end{cases}$$

Thus

$$f^{(n)}(0) = \begin{cases} (-1)^k, & \text{if } n = 2k \text{ is even} \\ 0, & \text{if } n = 2k-1 \text{ is odd} \end{cases}$$

Therefore the Taylor polynomial of $f(x)$ of degree $n = 2k$ at $x = 0$ is

$$\begin{aligned} p_{2k}(x) &= f(0) + \frac{f''(0)}{2!}x^2 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(6)}(0)}{6!}x^6 + \cdots + \frac{f^{(2k)}(0)}{(2k)!}x^{2k} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} \end{aligned}$$

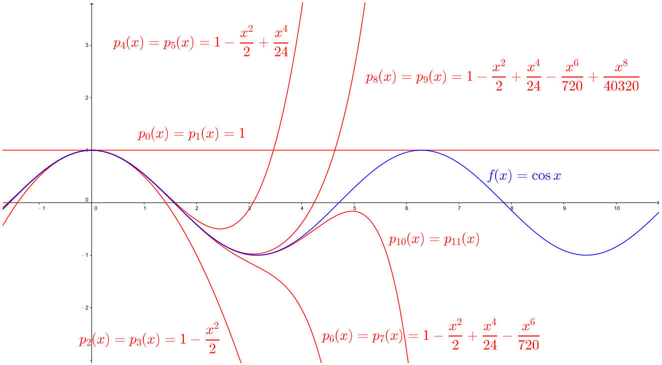

Figure: Taylor polynomials for $f(x) = \cos x$ at $x = 0$

Example

We are going to find the Taylor polynomial of $f(x) = \dfrac{1}{x}$ at $x = 1$. The k-th derivatives of $f(x)$ is

$$\frac{d^k}{dx^k}\frac{1}{x} = \frac{(-1)^k k!}{x^{k+1}}.$$

Thus

$$f^{(k)}(1) = (-1)^k k!.$$

Therefore the Taylor polynomial of $f(x)$ of degree n at $x = 1$ is

$$\begin{aligned} p_n(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \cdots + \frac{f^{(n)}(1)}{(n)!}(x-1)^n \\ &= 1 - (x-1) + \frac{2!(x-1)^2}{2!} - \frac{3!(x-1)^2}{3!} + \cdots + \frac{(-1)^n n!(x-1)^n}{n!} \\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n(x-1)^n \end{aligned}$$

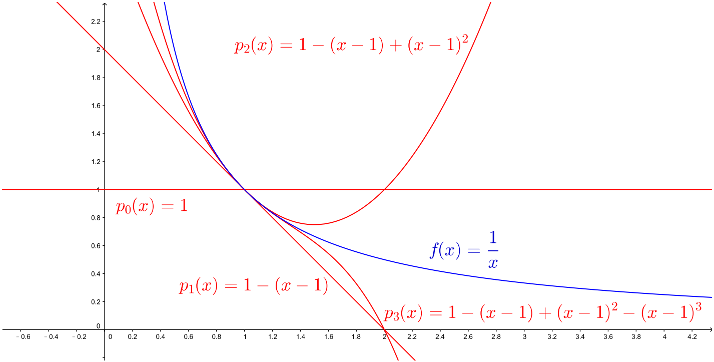

Figure: Taylor polynomials for $f(x) = \dfrac{1}{x}$ at $x = 1$

Example

We are going to find the Taylor polynomial of $f(x) = (1+x)^\alpha$ at $x = 0$, where $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} f^{(k)}(0) &= \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)(1+x)^{\alpha-k}|_{x=0} \\ &= \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1). \end{aligned}$$

Therefore the Taylor polynomial of $f(x)$ of degree n at $x = 0$ is

$$\begin{aligned} p_n(x) &= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f^{(3)}(0)x^3}{3!} + \cdots + \frac{f^{(n)}(0)x^n}{(n)!} \\ &= 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \cdots + \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)x^n}{n!} \\ &= \binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots + \binom{\alpha}{n}x^n \end{aligned}$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}.$$

Example

The following table shows the Taylor polynomials of degree n for $f(x)$ at $x=0$.

$f(x)$	Taylor polynomial
e^x	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}$
$\cos x$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+\frac{(-1)^k x^{2k}}{(2k)!},\ n=2k$
$\sin x$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+\frac{(-1)^k x^{2k+1}}{(2k+1)!},\ n=2k+1$
$\ln(1+x)$	$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+\frac{(-1)^{n+1}x^n}{n}$
$\frac{1}{1-x}$	$1+x+x^2+x^3+\cdots+x^n$
$\sqrt{1+x}$	$1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\cdots+\frac{(-1)^{n+1}(2n-3)!!x^n}{2^n n!}$
$(1+x)^\alpha$	$1+\alpha x+\frac{\alpha(\alpha-1)x^2}{2!}+\frac{\alpha(\alpha-1)(\alpha-2)x^3}{3!}+\cdots+\binom{\alpha}{n}x^n$

Example

The following table shows the Taylor polynomials of degree n for $f(x)$ at the given center.

$f(x)$	Taylor polynomial
$\cos x;\ x=\pi$	$-1+\frac{(x-\pi)^2}{2!}-\frac{(x-\pi)^4}{4!}+\cdots+\frac{(-1)^{k+1}(x-\pi)^{2k}}{(2k)!}$
$e^x;\ x=2$	$e^2+e^2(x-2)+\frac{e^2(x-2)^2}{2!}+\cdots+\frac{e^2(x-2)^n}{n!}$
$\frac{1}{x};\ x=1$	$1-(x-1)+(x-1)^2-(x-1)^3+\cdots+(-1)^n(x-1)^n$
$\frac{1}{2+x};\ x=0$	$\frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\cdots+\frac{(-1)^n x^n}{2^{n+1}}$
$\frac{1}{3-2x};\ x=1$	$1+2(x-1)+4(x-1)^2+8(x-1)^3+\cdots+2^n(x-1)^n$
$\sqrt{100-2x};\ x=0$	$10-\frac{x}{10}-\frac{x^2}{2000}-\frac{x^3}{200000}-\cdots-\frac{(2n-3)!!x^n}{10^{2n-1}n!}$

Theorem (Taylor's theorem)

Let $f(x)$ be a function such that the $n+1$-th derivative exists. Let $p_n(x)$ be the Taylor polynomial of degree n of $f(x)$ at $x=a$. Then for any x, there exists ξ between a and x such that

$$\begin{aligned} f(x) &= p_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \\ &= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}. \end{aligned}$$

Note: Taylor polynomial can be used to find the approximate value of a function for a given value of x. The Taylor's theorem tell us the possible values of the error, that is the difference between the approximated value $p_n(x)$ and the actual value $f(x)$.

Proof (Taylor's theorem)

First, suppose $f^{(k)}(a) = 0$ *for* $k = 0, 1, 2, \ldots, n$. *Then* $p_n(x) = 0$ *is the zero polynomial. Let* $g(x) = (x-a)^{n+1}$. *Observe that* $g^{(k)}(a) = 0$ *for* $k = 0, 1, 2, \ldots, n$ *and* $g^{(n+1)}(x) = (n+1)!$. *Applying Cauchy's mean value theorem successively, there exists* $\xi_1, \xi_2, \ldots, \xi = \xi_{n+1}$ *between* a *and* x *such that*

$$\begin{aligned}
\frac{f'(\xi_1)}{g'(\xi)} &= \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f(x)}{g(x)} \quad (f, g \text{ on } [a,x]) \\
\frac{f''(\xi_2)}{g''(\xi_2)} &= \frac{f'(\xi_1)-f'(a)}{g'(\xi_1)-g'(a)} = \frac{f'(\xi_1)}{g'(\xi_1)} = \frac{f(x)}{g(x)} \quad (f', g' \text{ on } [a,\xi_1]) \\
&\ \vdots \\
\frac{f^{(n+1)}(\xi)}{g^{(n+1)}(\xi)} &= \frac{f^{(n)}(\xi_n)-f^{(n)}(a)}{g^{(n)}(\xi_n)-g^{(n)}(a)} = \frac{f^{(n)}(\xi_n)}{g^{(n)}(\xi_n)} = \frac{f(x)}{g(x)} \quad (f^{(n)}, g^{(n)} \text{ on } [a,\xi_n])
\end{aligned}$$

Thus

$$f(x) = \frac{f^{(n+1)}(\xi)}{g^{(n+1)}(\xi)} g(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

Proof (Taylor's theorem).

For the general case, let

$$h(x) = f(x) - p_n(x).$$

Then $h^{(k)}(a) = 0$ for $k = 0, 1, 2, \ldots, n$ and $h^{(n+1)}(x) = f^{(n+1)}(x)$. Applying the first part of the proof to $h(x)$, there exists ξ between a and x such that

$$\begin{aligned} h(x) &= \frac{h^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \\ f(x) - p_n(x) &= \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \end{aligned}$$

as desired. □

Example

Let $f(x) = \cos x$.
The Taylor polynomial of degree 5 for $f(x)$ at $x = 0$ is

$$p_5(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

For any $|x| \leq 1.5$, we have

$$|\cos x - p_5(x)| = \frac{|f^{(6)}(\xi)|}{6!}(1.5)^6 \leq \frac{1.5^6}{6!} < 0.01583$$

The Taylor polynomial of degree 11 for $f(x)$ at $x = 0$ is

$$p_{11}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800}.$$

For any $|x| \leq 1.5$, we have

$$|\cos x - p_{11}(x)| = \frac{|f^{(12)}(\xi)|}{12!}(1.5)^{12} \leq \frac{1.5^{12}}{12!} < 2.71 \times 10^{-7}.$$

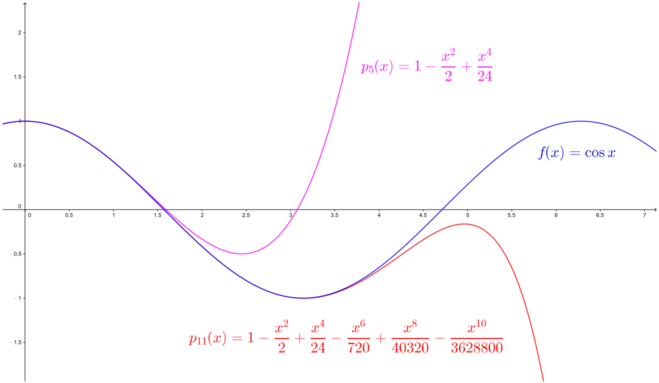

Figure: Taylor polynomials for $f(x) = \cos x$

Example

The following table shows the value of $p_n(x)$, the actual error which is difference $|\cos x - p_n(x)|$ and the largest possible error $\dfrac{x^{n+1}}{(n+1)!}$ for $x = 1.5$ and $x = 3$.

n	$x = 1.5$	Error	Largest	$x = 3$	Error	Largest
1	1	0.9292628	1.125	1	1.98999	4.5
3	-0.125	0.19574	0.21094	-3.5	2.51001	3.375
5	0.0859372	0.01521	0.01583	-0.125	0.86499	1.0125
7	0.0701172	6.21×10^{-4}	6.36×10^{-4}	-1.1375	0.14751	0.16273
9	0.0707528	1.57×10^{-5}	1.59×10^{-5}	-0.97478	0.01522	0.01628
11	0.0707369	2.68×10^{-7}	2.71×10^{-7}	-0.99105	0.00106	0.00111
cos	0.0707372			-0.98999		

Example

Let $f(x) = \ln(1+x)$. Then $f^{(n)}(x) = \dfrac{(-1)^{n+1}(n-1)!}{(1+x)^n}$ for $n \geq 1$.

The Taylor polynomial of degree n of $f(x)$ is

$$p_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n}.$$

Note that $f(1) = \ln 2$. By Taylor's theorem, there exists $0 < \xi < 1$ such that

$$|\ln 2 - p_n(1)| = \frac{|f^{(n+1)}(\xi)|}{(n+1)!} = \frac{1}{(n+1)(1+\xi)^{n+1}} < \frac{1}{n+1}.$$

When $n = 10,000$, we have $|\ln 2 - p_{10000}(1)| < \dfrac{1}{10001}$. As a matter of fact,

$$\begin{aligned} p_{10000}(1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{10000} &\approx 0.69309718 \\ \ln 2 &\approx 0.69314718 \end{aligned}$$

Example

$$f(x) = \ln(1+x);\ \ p_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n}.$$

For $x = 2$, by Taylor's theorem, there exists $0 < \xi < 2$ such that the error is

$$E_n = |\ln 3 - p_n(2)| = \frac{|f^{(n+1)}(\xi)|}{(n+1)!} \cdot 2^{n+1} = \frac{2^{n+1}}{(n+1)(1+\xi)^{n+1}}.$$

Note that $\frac{2^{n+1}}{(n+1)3^{n+1}} < E_n < \frac{2^{n+1}}{n+1}$. The table below shows the least possible, largest possible and actual values of the error E_n for various n.

n	$p_n(2)$	Least	Actual	Largest
5	5.06667	0.01463	3.96805	10.6667
10	-64.8254	0.00105	65.924	186.18
15	1424.42	9.52×10^{-5}	1423.33	4096
20	-34359.7	9.55×10^{-6}	34360.8	99864.4

The actual value is $f(2) = \ln(3) \approx 1.09861$. One can never get a good approximation of $\ln 3$ from $p_n(2)$ because $p_n(2)$ is divergent as $n \to \infty$.

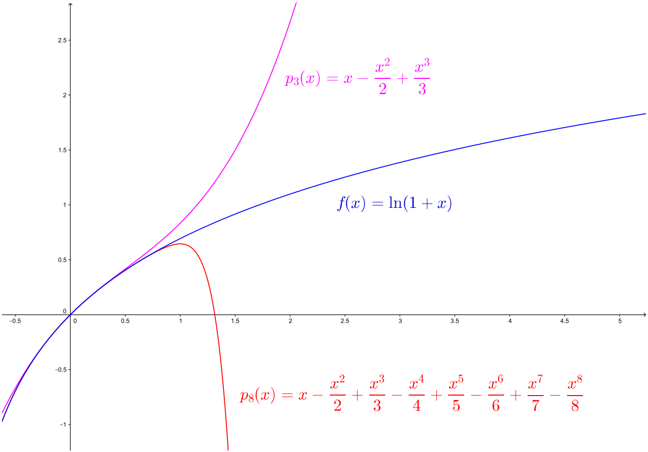

Figure: Taylor polynomials for $f(x) = \ln(1+x)$

Definition (Taylor series)

Let $f(x)$ be an infinitely differentiable function. The **Taylor series** of $f(x)$ at $x = a$ is the infinite power series

$$T(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots .$$

Example

The following table shows the Taylor series for $f(x)$ at the given center.

$f(x)$	Taylor series
$e^x;\ x=0$	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$
$\cos x;\ x=0$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$
$\sin x;\ x=\pi$	$-(x-\pi)+\frac{(x-\pi)^3}{3!}-\frac{(x-\pi)^5}{5!}+\cdots$
$\ln x;\ x=1$	$(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots$
$\sqrt{1+x};\ x=0$	$1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\cdots$
$\frac{1}{\sqrt{1+x}};\ x=0$	$1-\frac{x}{2}+\frac{3x^2}{8}-\frac{5x^3}{16}+\frac{35x^4}{128}-\frac{63x^5}{256}+\cdots$
$(1+x)^\alpha;\ x=0$	$1+\alpha x+\frac{\alpha(\alpha-1)x^2}{2!}+\frac{\alpha(\alpha-1)(\alpha-2)x^3}{3!}+\cdots$

Example

$f(x)$	Taylor series
e^x;	$\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$
$\cos x$;	$\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$
$\sin x$;	$\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$
$\ln(1+x)$;	$\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$
$\dfrac{1}{1-x}$;	$\displaystyle\sum_{k=0}^{\infty} x^k$
$(1+x)^\alpha$;	$\displaystyle\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k, \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}$
$\tan^{-1} x$;	$\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$
$\sin^{-1} x$;	$\displaystyle\sum_{k=0}^{\infty} \frac{(2k)! x^{2k+1}}{4^k (k!)^2 (2k+1)}$

Theorem

Suppose $T(x)$ is the Taylor series of $f(x)$ at $x = 0$. Then for any positive integer k, the Taylor series for $f(x^k)$ at $x = 0$ is $T(x^k)$.

Example

$f(x)$	Taylor series at $x = 0$
$\dfrac{1}{1+x^2}$	$1 - x^2 + x^4 - x^6 + \cdots$
$\dfrac{1}{\sqrt{1-x^2}}$	$1 + \dfrac{x^2}{2} + \dfrac{3x^4}{8} + \dfrac{5x^6}{16} + \dfrac{35x^8}{128} + \cdots$
$\dfrac{\sin x^2}{x^2}$	$1 - \dfrac{x^4}{3!} + \dfrac{x^8}{5!} - \dfrac{x^{12}}{7!} + \cdots$

Theorem

Suppose the Taylor series for $f(x)$ *at* $x = 0$ *is*

$$T(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Then the Taylor series for $f'(x)$ *is*

$$T'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots .$$

Example

Find the Taylor series of the following functions.

1. $\dfrac{1}{(1+x)^2}$
2. $\tan^{-1} x$

Solution

1. *Let* $F(x) = -\dfrac{1}{1+x}$ *so that* $F'(x) = \dfrac{1}{(1+x)^2}$. *The Taylor series for* $F(x)$ *at* $x = 0$ *is*

$$T(x) = -1 + x - x^2 + x^3 - x^4 + \cdots.$$

Therefore the Taylor series for $F'(x) = \dfrac{1}{(1+x)^2}$ *is*

$$T'(x) = 1 - 2x + 3x^2 - 4x^3 + \cdots.$$

Solution

2. *Suppose the Taylor series for* $f(x) = \tan^{-1} x$ *at* $x = 0$ *is*

$$T(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \cdots .$$

Now comparing $T'(x)$ *with the Taylor series for* $f'(x) = \dfrac{1}{1+x^2}$ *which takes the form*

$$1 - x^2 + x^4 - x^6 + \cdots ,$$

we obtain the values of $a_1, a_2, a_3, \ldots$ *and get*

$$T(x) = a_0 + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

Since $a_0 = T(0) = f(0) = 0$, *we have*

$$T(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

Theorem

Suppose the Taylor series for $f(x)$ and $g(x)$ at $x = 0$ are

$$\begin{aligned} S(x) &= \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots, \\ T(x) &= \sum_{k=0}^{\infty} b_k x^k = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots, \end{aligned}$$

respectively. Then the Taylor series for $f(x)g(x)$ at $x = 0$ is

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n \\ = \quad & a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots \end{aligned}$$

Proof.

The coefficient of x^n of the Taylor series of $f(x)g(x)$ at $x = 0$ is

$$\begin{aligned}\frac{(fg)^{(n)}(0)}{n!} &= \sum_{k=0}^{n} \binom{n}{k} \frac{f^{(k)}(0)g^{(n-k)}(0)}{n!} \quad \text{(Leibniz's formula)}\\ &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot \frac{f^{(k)}(0)g^{(n-k)}(0)}{n!}\\ &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot \frac{g^{(n-k)}(0)}{(n-k)!}\\ &= \sum_{k=0}^{n} a_k b_{n-k}\end{aligned}$$

□

Example

1. The Taylor series for $e^{4x}\ln(1+x)$ is

$$\begin{aligned} & \left(1+4x+\frac{16x^2}{2!}+\frac{64x^3}{3!}+\cdots\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right) \\ = \quad & x+\left(-\frac{1}{2}+4\right)x^2+\left(\frac{1}{3}+4\cdot\left(-\frac{1}{2}\right)+8\right)x^3+\cdots \\ = \quad & x+\frac{7x^2}{2}+\frac{19x^3}{3}+\cdots \end{aligned}$$

2. The Taylor series for $\dfrac{\tan^{-1}x}{\sqrt{1-x^2}}$ is

$$\begin{aligned} & \left(x-\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right)\left(1+\frac{x^2}{2}+\frac{3x^4}{8}+\cdots\right) \\ = \quad & x+\left(\frac{1}{2}-\frac{1}{3}\right)x^3+\left(\frac{3}{4}-\frac{1}{3}\cdot\frac{1}{2}+\frac{1}{5}\right)x^5+\cdots \\ = \quad & x+\frac{x^3}{6}+\frac{49x^5}{120}+\cdots \end{aligned}$$

Theorem

For any power series

$$S(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

there exists $R \in [0, +\infty]$ called radius of convergence such that

1. *$S(x)$ is absolutely convergent for any $|x| < R$ and divergent for any $|x| > R$. For $|x| = R$, $S(x)$ may or may not be convergent.*
2. *When $S(x)$ is considered as a function of x, it is differentiable on $(-R, R)$ and its derivative is*

$$S'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots.$$

Caution! There exists R such that the Taylor series $T(x)$ is convergent when $|x| < R$. Although in most examples, $T(x)$ converges to $f(x)$ when it is convergent, there are examples that $T(x)$ does not converge to $f(x)$.

Example

The following table shows the convergence of Taylor series of various functions.

$f(x)$	$T(x)$	R	$x=-R$	$x=R$
e^x	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$	$+\infty$	Not Applicable	Not Applicable
$\cos x$	$1-x^2+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$	$+\infty$	Not Applicable	Not Applicable
$\sin x$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$	$+\infty$	Not Applicable	Not Applicable
$\ln(1+x)$	$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$	1	Divergent	$\ln 2$
$\sqrt{1+x}$	$1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\cdots$	1	0	$\sqrt{2}$
$\frac{1}{1+x^2}$	$1-x^2+x^4-x^6+\cdots$	1	Divergent	Divergent
$\tan x$	$x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\cdots$	$\frac{\pi}{2}$	Divergent	Divergent
$\tan^{-1} x$	$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$	1	$-\frac{\pi}{4}$	$\frac{\pi}{4}$

Question

Let $T(x)$ be the Taylor series of a function $f(x)$ at $x = a$. Does $T(x)$ always converge to $f(x)$ at the points where $T(x)$ is convergent?

Answer

No. There exists function $f(x)$ with Taylor series $T(x)$ at $x = a$ such that

1. *$T(x)$ is convergent for any real number $x \in \mathbb{R}$, and*
2. *$T(x)$ does not converge to $f(x)$ for any $x \neq a$.*

Theorem

Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

Then the Taylor series for $f(x)$ *at* $x = 0$ *is* $T(x) = 0$.

Note. It is obvious that $f(x) \neq 0$ when $x \neq 0$. Therefore $T(x) \neq f(x)$ for any $x \neq 0$.

Proof.

We claim that for any nonnegative integer n, we have

$$f^{(n)}(x) = \begin{cases} \dfrac{P_n(x)}{x^{3n}} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

for some polynomial $P_n(x)$. In particular, $f^{(n)}(0) = 0$ for any $n = 0, 1, 2, \cdots$ which implies that $T(x) = 0$. We prove that claim by induction on n. When $n = 0$, $f^{(0)}(x) = f(x)$ and there is nothing to prove. Suppose the claim is true for $n = k$. Then when $x \neq 0$,

$$f^{(k+1)} = \frac{x^{3k}(P_k' + \frac{2P_k}{x^3}) - 3kx^{3k-1}P_k}{x^{6k}} e^{-\frac{1}{x^2}} = \frac{x^3 P_k' - 3kx^2 P_k + 2P_k}{x^{3(k+1)}} e^{-\frac{1}{x^2}}.$$

We may take $P_{k+1} = x^3 P_k' - 3kx^2 P_k + 2P_k$. On the other hand,

$$f^{(k+1)}(0) = \lim_{h \to 0} \frac{f^{(k)}(h) - f^{(k)}(0)}{h} = \lim_{h \to 0} \frac{P_k(h)}{h^{3k}} e^{-\frac{1}{h^2}} = \lim_{y \to +\infty} \frac{y^{3k} P_k(\frac{1}{y})}{e^{y^2}} = 0.$$

This completes the induction step and the proof of the claim □

Example

$$
\begin{aligned}
f'(x) &= \frac{2}{x^3} e^{-\frac{1}{x^2}} \\
f''(x) &= \frac{-6x^2+4}{x^6} e^{-\frac{1}{x^2}} \\
f^{(3)}(x) &= \frac{24x^4-36x^2+8}{x^9} e^{-\frac{1}{x^2}} \\
f^{(4)}(x) &= \frac{-120x^6+300x^4-144x^2+16}{x^{12}} e^{-\frac{1}{x^2}} \\
f^{(5)}(x) &= \frac{720x^8-2640x^6+2040x^4-480x^2+32}{x^{15}} e^{-\frac{1}{x^2}}
\end{aligned}
$$

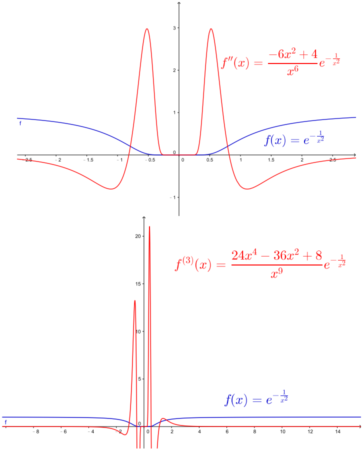

Figure: $f(x) = e^{-\frac{1}{x^2}}$