THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 3

Definition (Convergence of infinite series)

Let $\sum_{n=1}^{\infty} a_n$ be a infinite series. Then it is convergent if the sequence of **partial sum**

$$s_n = \sum_{k=1}^{n} a_k$$

is convergent. Then we define

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

Some theorems about infinite series

1. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$
2. ***Comparison test***
   If $0 \le |a_n| \le b_n$ for all n and $\sum_{n=0}^{\infty} b_n$ is convergent, then $\sum_{n=0}^{\infty} a_n$ is convergent
3. ***Alternating series test***
   If $\{a_n\}$ is a decreasing sequence of positive real numbers and $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=0}^{\infty} (-1)^n a_n$ is convergent.

Definition (Limit of functions)

Let $f(x)$ be a real-valued function.

1. $L \in \mathbb{R}$ is said to be a limit of $f(x)$ at $x = a$ if for any $\epsilon > 0$, there exists δ such that

   $$\text{if } 0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon$$

   Here we write $\lim_{x\to a} f(x) = L$
2. $L \in \mathbb{R}$ is said to be a limit of $f(x)$ at $x = +\infty$ if for any $\epsilon > 0$, there exists $R \in \mathbb{R}$ such that

   $$\text{if } x > R \text{ then } |f(x) - L| < \epsilon$$

   Here we write $\lim_{x\to\infty} f(x) = L$

Remark: $f(a)$ may not exist even when $\lim_{x\to a} f(x)$ exists.

Problems that may be demonstrated in class :

Given that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, and $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Q1. Are the following infinite series convergent? Prove it.

(a) $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^4}$

(b) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$

(c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-2}}$

(d) $\sum_{n=1}^{\infty} \frac{4^n}{3^n+1}$

(e) $\sum_{n=2}^{\infty} \frac{n}{\ln n}$

(f) $\sum_{n=1}^{\infty} (-1)^n$

Q2. By using comparison test, prove the following statement: If $\sum_{n=1}^{\infty} a_n$ with $a_n > 0$ is convergent, then $\sum_{n=1}^{\infty} a_n^2$ is convergent.

Q3. (a) If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and (b_n) is a bounded sequence, show that $\sum_{n=1}^{\infty} a_n b_n$ is absolutely convergent.

(b) Give an example such that the above statement is false if *absolutely convergent* is replaced by *convergent*.

Q4. Compute the following limits:

(a) $\lim_{x\to 1} x+1$

(b) $\lim_{x\to 1} \frac{x^2-1}{x-1}$

(c) $\lim_{x\to 0} \frac{e^x-1}{x}$

(d) $\lim_{x\to\infty} \frac{6e^{4x}-e^{-2x}}{8e^{5x}-e^{2x}+3e^{-x}}$

(e) $\lim_{x\to\infty} \frac{3x^2+7x+5}{5x^2+2}$

(f) $\lim_{x\to\infty} x-\sqrt{x^2+x}$

(g) $\lim_{x\to 0} x\sin\frac{1}{x}$

(h) $\lim_{x \to 1} \frac{x^5 - 1}{x - 1}$

Q5. (a) Let $a \in \mathbb{R}$. Show that if $\lim_{x \to a} f(x)$ exists, then $\lim_{x \to a} [f(x)]^2$ exist.

(b) Is the converse true? Prove or disprove.