THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2230A (First term, 2015–2016)
Complex Variables and Applications
Notes 14 Maximum Modulus

Before, it is proved that $|f|$ cannot be a maximum at the center of a ball if f is analytic. Indeed, even when the domain is not a ball, but a general Ω, such situation that $|f|$ attains a maximum at some $z_0 \in \Omega$ imposes very strong restriction on f.

Theorem 14.1 (Maximum Modulus Principle). *Let f be an analytic function on a domain Ω. If there is a $z_0 \in \Omega$ such that*

$$|f(z_0)| = \sup\{|f(\zeta)| : \zeta \in \Omega\},$$

then f is a constant function on Ω.

A topological proof. Readers may need some basic knowledge about connectedness in topology to understand this proof.

Let $W = \{ z \in \Omega : |f(z)| = |f(z_0)| \}$. We will prove that both W and $\Omega \setminus W$ are open sets. Then by connectedness of Ω, it follows that $W = \Omega$. Further by an exercise about Cauchy-Riemann Equations, that $|f|$ is constant will lead to that f is constant.

First, let us show that $\Omega \setminus W$ is open. Let $z \notin W$, then $|f(z)| < |f(z_0)|$. By continuity of $|f|$, there is $\varepsilon > 0$ such that for each $\zeta \in B(z, \varepsilon) \subset \Omega$, $|f(\zeta)| < |f(z_0)|$. Thus, $z \in B(z, \varepsilon) \subset \Omega \setminus W$.

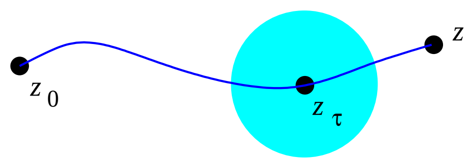
Second, we show that W is open. Let $z \in W$, then take a $\delta > 0$ such that $B(z, \delta) \subset \Omega$. Now,

$$|f(z)| = |f(z_0)| \geq \max\{ |f(\zeta)| : \zeta \in \partial B(z, \delta) \}.$$

Using Proposition 14.1 above, f is constant on $B(z, \delta)$ and so $B(z, \delta) \subset W$. $\square$

An analytical proof. Here is a proof that uses more analytical techniques. The topological nature is hidden.

Given any $z \in \Omega$, take a path $\gamma : [0, 1] \to \Omega$ joining $z_0 = \gamma(0)$ to $z = \gamma(1)$. This has subtly used the open and connectedness of Ω. We try to show that $|f(\zeta)| = |f(z_0)|$ for all $\zeta \in \gamma$. Essentially, let $T = \{ t \in [0, 1] : |f(\gamma(t))| = |f(z_0)| \}$ and $\tau = \sup T$. We will show that $\tau = 1$.



Let $z_\tau = \gamma(\tau)$. Then $|f(z_\tau)| = |f(z_0)|$ by continuity. Since $z_\tau \in \Omega$, there is $\varepsilon > 0$ such that the closed disk $D = \overline{B(z_\tau, \varepsilon)} \subset \Omega$. Then $|f(z_0)| = \max\{ |f(\zeta)| : \zeta \in \partial D \}$ and by Proposition 14.1, f is constant on D. Thus, the disk D must contain $\gamma(t)$ with $t \geq \tau$ and $\tau = 1$. $\square$

Combining the results of Corollary 14.2 and Maximum Modulus Principle, we may imagine how the $|f|$ looks like on a domain Ω. Starting from any point inside Ω, the value of $|f|$ must only "increase" at least in some direction. Thus, it must be "largest" at somewhere on the boundary, if Ω has a boundary. Of course, Ω may not have a boundary and in that case, $|f|$ may increase to infinity.

COROLLARY 14.2. *Let $K \subset \mathbb{C}$ be a closed and bounded set and $K \subset \Omega$ such that f is analytic on the domain Ω. Then*

$$\max \{ |f(z)| : \ z \in K \} = \max \{ |f(\zeta)| : \ \zeta \in \partial K \} .$$

That is, the maximum of $|f|$ is attained somewhere on the boundary.

Note that this corollary has already included the case that f is a constant function. Moreover, if f is analytic on a closed ball $D = \overline{B(z_0, R)}$, then for any $r \le R$, we have

$$M_r = \max \{ |f(z)| : \ |z - z_0| = r \} \le M_R = \max \{ |f(\zeta)| : \ |\zeta - z_0| = R \} .$$

This refers to our previous discussion that $\max |f|$ along concentric circles is increasing wrt the radius. For non-constant analytic function, it should be strictly increasing.

EXAMPLE 14.3. Let $f(z) = z^2 - z$. Find the maximum of $|f|$ on $B(0, R)$ and at which point z the maximum is attained.

Since f is analytic on $\mathbb{C}$, by Maximum Modulus Principle,

$$\max \{ |f(z)| : \ z \in B(0, R) \} = \max \{ |f(z)| : \ |z| = R \}$$

Therefore, in polar form, $\max |f(z)| = \left| f(Re^{\mathbf{i}\theta}) \right|$ for some θ.

Note that $f(Re^{\mathbf{i}\theta}) = (R^2 \cos 2\theta - R \cos \theta) + \mathbf{i}((R^2 \sin 2\theta - R \sin \theta)$, we have

$$\left| f(Re^{\mathbf{i}\theta}) \right|^2 = R^2 \left[R^2 - 2R \cos \theta + 1 \right] .$$

The maximum of $\left| f(Re^{\mathbf{i}\theta}) \right|^2$ will be attained when $\cos \theta = -1$, i.e., at $\theta = \pi$. Thus,

$$\max \{ |f(z)| : \ z \in B(0, R) \} = R(R + 1) = |f(-R)| .$$

14.1 "Converse" of Cauchy-Goursat

It has already been seen that analytic functions possess interesting and surprising properties. It can be differentiate infinitely many times even the original assumption is only once. Its value at a point is the average value of point circulating it. There are many other properties that form the heart of studying complex functions.

In the Cauchy-Goursat Theorem, we usually have a fixed simple closed contour Γ and its bounded complement component S_b such that $\Gamma \cup S_b \subset \Omega$ and f is assumed to be analytic on Ω. The "converse" deals with integrals over arbitrary contours and a fixed domain Ω to conclude that f is more than analytic.

THEOREM 14.4 (Morera Theorem). *Let Ω be a domain (no topological condition) and $f : \ \Omega \to \mathbb{C}$ be a continuous function satisfying that for all simple closed contour Γ,*

$$\int_\Gamma f(z)\, dz = 0 ,$$

then f has an antiderivative and so is analytic on Ω.

Proof. In order to prove this result, one needs to construct an antiderivative F, i.e., $F'(z) = f(z)$ for all $z \in \Omega$. Since antiderivative is up to a constant, there is always a choice. Take $z_0 \in \Omega$ and we will define $F(z)$ for any $z \in \Omega$ by contour integration. Take a contour γ from z_0 to z (it exists because Ω is open connected) and define

$$F(z) = \int_\gamma f(\zeta)\, d\zeta\,.$$

The value of $F(z)$ is independent of the choice of γ because of the zero integral assumption. What remains is to show that $F'(z) = f(z)$. For a sufficiently small $h \in \mathbb{C}$, both z and $z+h$ lie in Ω and they can be joined by straight line L. Choose a contour γ from z_0 to z, then $\Gamma = (\gamma, L)$ is a contour from z_0 to $z+h$. Thus,

$$\begin{aligned}\frac{1}{h}\left[\, F(z+h) - F(z)\,\right] &= \frac{1}{h}\left[\int_\Gamma f(\zeta)\, d\zeta - \int_\gamma f(\zeta)\, d\zeta\right] \\ &= \frac{1}{h}\int_L f(\zeta)\, d\zeta = \frac{1}{h}\int_0^1 f(z+th)\, h\, dt = \int_0^1 f(z+th)\, dt\,.\end{aligned}$$

Apply the continuity of f on L and take limit $h \to 0$ to have $F'(z) = f(z)$. $\square$

EXAMPLE 14.5. Consider the function $f(z) = 1/z^3$ and $g(z) = 1/z$ on $\mathbb{C} \setminus \{0\}$. The crucial contour to consider is a circle C with center 0, by direct calculation,

$$\int_C \frac{1}{z^3}\, dz = 0 \qquad \text{while} \qquad \int_C \frac{1}{z}\, dz = 2\pi\mathbf{i} \neq 0\,.$$

This is the fundamental reason why f has anti-derivatives on $\mathbb{C} \setminus \{0\}$ but not g while both functions are analytic on the domain.