THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2230A (First term, 2015–2016)
Complex Variables and Applications
Notes 8 Legacy of Logarithm

8.1 Complex Powers

It is helpful to recall our previous knowledge of powers, x^p where $x \in \mathbb{R}$. It is easy to define x^n for a positive integer n, i.e., repeated multiplying x for n times. Then, $x^{1/n}$ is defined to be "the" number that $\left(x^{1/n}\right)^n = x$. This involves the existence of such number and so it is not defined for even n and $x < 0$. Nevertheless, for suitable x, we are able to define rational powers, $x^{m/n}$. When it comes to any powers, i.e., x^p for $p \in \mathbb{R}$, more issues arisen. The most logical way is to define x^p using logarithm, i.e., $x^p = e^{p \log x}$. However, $\log x$ has no meaning for $x < 0$ while in many cases, x^p still makes sense for $x < 0$.

Recall that in the context of complex, logarithm is no longer a number, but a set,

$$\log z = \ln|z| + \mathbf{i}\arg(z)\,.$$

Then the definition of powers will be further complicated. For a positive integer n, we wish that the new definition of ζ^n is consistent with our intuition of repeated multiplication. When it comes to $p/q \in \mathbb{Q}$, $\zeta^{p/q}$ becomes more complicated because there will be q values. The set $\log\zeta$ may be indeed good for doing this. Let us first look at a fallacy.

Theorem 8.1 (False). *There is no complex number.*

Proof. It only needs to prove that $\mathbf{i}$ is indeed a real number.

$$\mathbf{i} = \sqrt{-1} = (-1)^{1/2} = (-1)^{2/4} = \sqrt[4]{(-1)^2} = \sqrt[4]{1} = \pm 1 \in \mathbb{R}\,.$$

Since $\mathbf{i} \in \mathbb{R}$, then $a + \mathbf{i}b \in \mathbb{R}$ for all $a, b \in \mathbb{R}$ and $\mathbb{C} \subset \mathbb{R}$. ☐

With this example, one may expect there will be more problems when ζ^c with $c \in \mathbb{R}$ or $c \in \mathbb{C}$ is involved. The crucial point to make things correct is by always working on sets.

Let $\zeta, c \in \mathbb{C}$, we define $\boxed{\zeta^c \overset{\text{def}}{:=} e^{c\log\zeta},}$ which is a set of complex numbers. More precisely,

$$\zeta^c = \left\{ e^{c\ln|\zeta| + \mathbf{i}(c\theta)} :\ \theta \in \arg\zeta \right\}.$$

Note that if $n \in \mathbb{N}$ and $\zeta = |\zeta|\, e^{\mathbf{i}\theta}$ for some particular $\theta \in \arg\zeta$, then the set $n\mathbf{i}\arg(\zeta)$ contains $n(\theta + 2k\pi\mathbf{i})$ and $e^{\mathbf{i}n(\theta+2k\pi)} = 1$. Therefore, only a single value $|\zeta|^n$ occurs in the set ζ^n. Similarly, in finding $\zeta^{1/n}$, though there are infinitely many values in $\mathbf{i}\arg(\zeta)/n$, the set $\zeta^{1/n}$ only contains n values, namely,

$$\zeta^{1/n} = \left\{ |\zeta|^{1/n}\, e^{\mathbf{i}(\theta/n + 2k\pi/n)} :\ k = 0, 1, 2, \ldots, n-1 \right\}.$$

In general, the set ζ^c is an infinite set. The choice of values in $\log\zeta$ indeed comes down to a choice of values in $\arg(\zeta)$. One has to be consistent in making the choice. For example, in the calculation below

$$\begin{aligned}\zeta^{a+\mathbf{i}b} &= e^{(a+\mathbf{i}b)\log\zeta} = e^{(a+\mathbf{i}b)(\ln|\zeta|+\mathbf{i}\arg(\zeta))} \\ &= e^{a\ln|\zeta|-b\arg(\zeta)}\cdot e^{\mathbf{i}(b\ln|\zeta|+a\arg(\zeta))},\end{aligned}$$

one must pick the same value in the two places of $\arg(\zeta)$ to have a value for the power.

8.1.1 Choices and Branches

Let us use the fallacy above to illustrate how complex powers must be handled carefully.

Exercise 8.2. Carefully work according to the definition to show that

$$(-1)^{1/2} = \{\,\mathbf{i}, -\mathbf{i}\,\} = (-1)^{2/4}, \qquad (1)^{1/4} = \{\,1, \mathbf{i}, -1, -\mathbf{i}\,\}\ .$$

From the above exercise, you may see that $\zeta^{p/q} \subset (\zeta^p)^{1/q}$ and equality holds only if p, q are relatively prime. When such trouble already exists for a rational power, one should see the need of working carefully when irrational or complex powers are involved; because in those cases, we are dealing with infinite sets.

The function notation $z \in \Omega \ \mapsto\ z^c$ is indeed a set value function. First, it is defined only if $0 \notin \Omega$. Second, a continuous choice of the function comes from that of $\log z$, thus $\Omega \neq \mathbb{C}\setminus\{0\}$. For example, the *principal branch* of z^c is $\exp(c\,\mathrm{Log}\,z)$ defined on $\mathbb{C}\setminus H_{-\pi}$, i.e., the nonpositive real axis $(-\infty, 0]$ is removed.

Many well-known facts about powers and indices must be interpreted as sets. For example,

$$z^{c_1+c_2} = z^{c_1}\cdot z^{c_2} \quad \text{and} \quad (z_1z_2)^c = z_1^c\cdot z_2^c$$

should somewhat be seen as consequences of $\log(z_1z_2) = \log(z_1) + \log(z_2)$.

Finally, similar to the situation of $\log z$, if $f(z)$ is a continuous branch of z^c on a domain Ω, then f is automatically analytic on Ω. For simplicity of notation, we often write

$$\frac{\mathrm{d}}{\mathrm{d}z}(z^c) = c\,z^{c-1},$$

which again should be interpreted as both sides of the equation are taking the same branch of $\log z$.

`..... to be continued`