THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2230A (First term, 2015–2016)
Complex Variables and Applications
Notes 6 Elementary Functions

6.1 Exponential Function

We have ***defined*** the *exponential function* by the following,

$$\exp(z) = e^z = e^{x+\mathbf{i}y} :\stackrel{\text{def}}{=\!=} e^x \cos y + \mathbf{i} e^x \sin y\,.$$

There are several ***properties*** easily obtained as consequences from this definition.

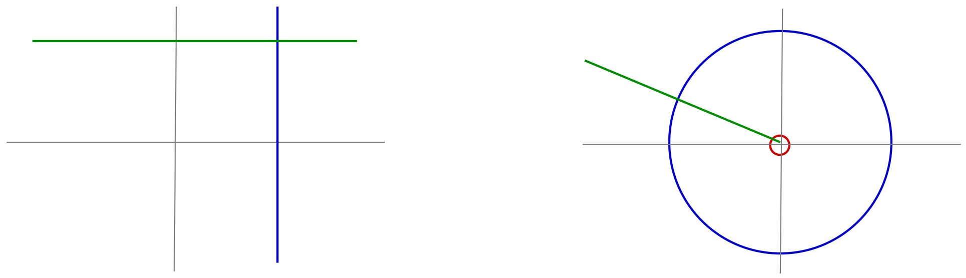
1. For $x \in \mathbb{R}$, $\exp(x) = e^{x+\mathbf{i}0} = e^x$. So, it coincides with the real function.
2. For $y \in \mathbb{R}$, $\exp(\mathbf{i}y) = e^{0+\mathbf{i}y} = \cos y + \mathbf{i}\sin y$.
3. Then obviously, $e^x \cdot e^{\mathbf{i}y} = e^{x+\mathbf{i}y}$, which behaves like the index law. We can further prove
$$e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}, \qquad \text{for all } z_1, z_2 \in \mathbb{C}.$$
4. Similar to real case, $e^z \neq 0$ for all $z \in \mathbb{C}$. This follows easily from $|e^z| = e^x$.
5. As in the real case, $\exp(0) = e^0 = 1$. For real variable, this is the only case because $x \mapsto e^x$ is one-to-one. However, for $z \in \mathbb{C}$, $e^z = 1$ if and only if $z = 2n\pi\mathbf{i}$ for $n \in \mathbb{Z}$.
6. The above actually indicates that $z \in \mathbb{C} \mapsto e^z$ is NOT one-to-one, unlike the real variable. Since $\exp(z) = \exp(z + 2\pi\mathbf{i})$, we say that it has a *period* of $2\pi\mathbf{i}$.

6.1.1 Visualization

With the above properties, we can easily visualize the action of $z \mapsto e^z$. Observe the definition

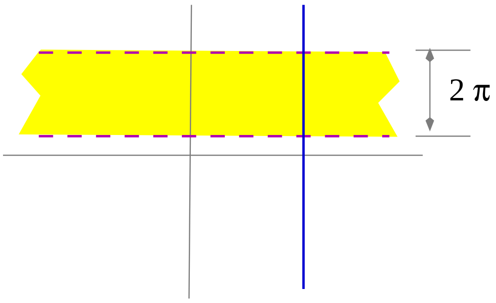
$$\underbrace{x + \mathbf{i}y}_{\text{rectangular form}} \mapsto \underbrace{e^x\left(\cos y + \mathbf{i}\sin y\right)}_{\text{this side becomes polar form, } r = e^x \text{ and } \theta = y.},$$

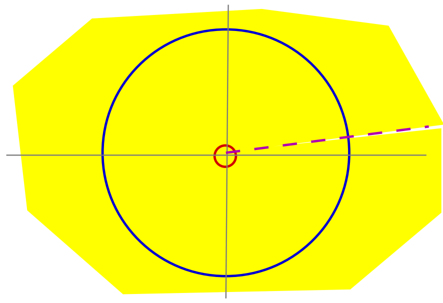
From that we immediately get the following. Note that the origin will never be reached in the target. A vertical line (with $x = a$) will be mapped to a circle (with radius e^a) while a horizontal line (with $y = \beta$) will be mapped to a radial line at an angle β.



EXERCISE 6.1. Imagine the vertical blue line moves from the left ($-\infty$) to the right (∞). How will its image circle changes? Do similar imagination for the horizontal green line.

As we have mentioned above, $z \mapsto e^z$ is not one-to-one. But obviously, the vertical lines are mapped to concentric circles in a one-to-one fashion. The non-injective situation occurs on the horizontal lines. In fact, for any horizontal strip of width 2π, the image is the whole $\mathbb{C} \setminus \{0\}$.





Exercise 6.2. Try to visualize the pre-images of vertical and horizontal lines. This is non-trivial.

6.2 Elementary Functions

Our attention are on analytic functions. Although there may be many of them, most of the examples are built up from some well-known functions. They are

> constant functions, polynomials, rational functions (fractions of two polynomials), and the exponential function.

From the knowledge of real variable, we know that there should also be

> trignonometric functions, hyperbolic functions, and finally the logarithm.

All of them are suitably defined for complex variable and they are called *elementary functions*.

6.2.1 Trigonometric and Hyperbolic Functions

There are two major objectives when we try to define them for $z \in \mathbb{C}$. First, they are the same function as in the real case when we put in $x + \mathbf{i}0$. Second, we would like to have most of the important properties of the functions.

Observe from the definition of $\exp(z) = \exp(x + \mathbf{i}y) = e^x \cos y + \mathbf{i}e^x \sin y$, we see that for $y \in \mathbb{R}$, both $\cos y$ and $\sin y$ can be expressed in terms of $\exp(0 + \mathbf{i}y) = e^{\mathbf{i}y}$. In fact,

$$\cos y = \frac{1}{2}\left(e^{\mathbf{i}y} + e^{-\mathbf{i}y}\right), \qquad \sin y = \frac{1}{2\mathbf{i}}\left(e^{\mathbf{i}y} - e^{-\mathbf{i}y}\right) \qquad \text{for } y \in \mathbb{R}\,.$$

Thus, the natural way to define $\cos z$ and $\sin z$ is simply replace the y in above by z. That is,

$$\cos z :\stackrel{\text{def}}{=\!=} \frac{1}{2}\left(e^{\mathbf{i}z} + e^{-\mathbf{i}z}\right), \qquad \sin z :\stackrel{\text{def}}{=\!=} \frac{1}{2\mathbf{i}}\left(e^{\mathbf{i}z} - e^{-\mathbf{i}z}\right)\,.$$

It achieves the first objective and makes sure that they are generalizations for the real variable functions. Secondly most of the properties of $\cos z$ and $\sin z$ can be obtained as consequences of the definition. To name a few,

1. Both $\sin z$ and $\cos z$ has a period of 2π.

2. The identity $\sin^2 z + \cos^2 z = 1$ holds for all $z \in \mathbb{C}$.

3. The index law of e^z actually leads to all the compound angle formulas for $\sin z$ and $\cos z$.

4. ***Warning***. There exists $z \in \mathbb{C}$ such that $|\sin z| \not\leq 1$ or $|\cos z| \not\leq 1$.

5. It is easy to see that both $\cos z$ and $\sin z$ are entire functions. For example,

$$\cos(x + \mathbf{i}y) = \frac{1}{2}\left(e^y + e^{-y}\right)\cos x + \frac{-\mathbf{i}}{2}\left(e^y - e^{-y}\right)\sin x\,.$$

   It can be seen that both the real and imaginary parts are C^∞ and satisfy the Cauchy-Riemann equations.

6. Then, obviously, $\frac{\mathrm{d}}{\mathrm{d}z}\sin z = \cos z$ and $\frac{\mathrm{d}}{\mathrm{d}z}\cos z = -\sin z$.

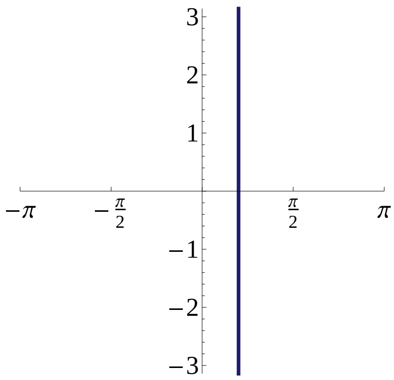
The other trigonometric functions, $\tan z$, $\sec z$, etc., will be defined by $\sin z$ and $\cos z$. Similarly, we have hyperbolic functions, namely,

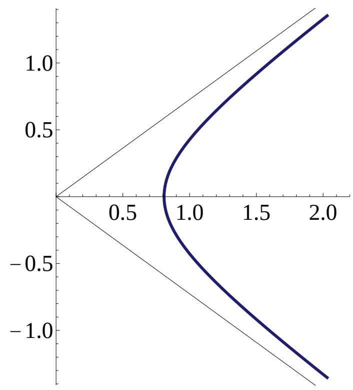
$$\cosh z :\stackrel{\text{def}}{=} \frac{1}{2}\left(e^z + e^{-z}\right), \qquad \sinh z :\stackrel{\text{def}}{=} \frac{1}{2}\left(e^z - e^{-z}\right).$$

Equivalently, $\cosh(z) = \cos(\mathbf{i}z)$ and $\sinh(z) = -\mathbf{i}\sin(\mathbf{i}z)$. Now, we see that in $\mathbb{R}$, there is no easy transformation that relates trigonometric functions to hyperbolic functions. But, when they are living in $\mathbb{C}$, they are indeed very close to each other. The properties of hyperbolic functions are obvious and we do not repeat them here.

6.2.2 Visualization

We will only visualize the function $z \mapsto \cos z$ as an exercise. The calculations and the image of the horizontal lines have similarly worked out in the example of $\sin z$ in Notes 02.





One should similarly work out the visualization for $\cosh z$ and $\sinh z$.