THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2230A (First term, 2015–2016)
Complex Variables and Applications
Notes 5 Analytic Functions

5.1 Working Principles

Previously, we have discussed complex differentiable functions from several viewpoints. But, it is convenient to have some simple principles for calculation.

5.1.1 Differenting Rules

Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be a complex function and $z_0 \in \Omega$. By definition, we have the complex derivative

$$f'(z_0) :\stackrel{\text{def}}{=\!=} \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} .$$

The first task is that we try not to use limit to find the complex derivative unless it is unavoidable.

Note that this definition is FORMALLY the same as the one for a function from $\mathbb{R}$ to $\mathbb{R}$. Moreover, in proving the differentiation rules in that case, only properties about $+, -, \times, \div$ and limits are used. Therefore, we can conclude that *all the differentiation rules, such as* linearity, product rule, quotient rule, chain rule *will still be* VALID *for complex functions.* For example,

$$f(z) = \frac{1}{z}, \quad z \in \mathbb{C} \setminus \{0\}; \qquad \text{then, by Quotient Rule,} \qquad f'(z) = \frac{-1}{z^2} .$$

5.1.2 Using Real and Imaginary Parts

Very often, the function is given in the form $f(x + \mathbf{i}y) = u(x, y) + \mathbf{i}v(x, y)$, where $x, y, u, v \in \mathbb{R}$. In that case, we would like to find $f'(x + \mathbf{i}y)$ in terms of $u(x, y)$ and $v(x, y)$. Note that

$$f(z) = f(z_0) + f'(z_0)\,(z - z_0) + \varepsilon, \qquad \text{where } \frac{\varepsilon}{|z - z_0|} \to 0 .$$

If $f'(z_0)$ exists, then the corresponding differential matrix of $(x, y) \mapsto (u, v)$ is given by

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}_{(x_0,y_0)} = \begin{pmatrix} \operatorname{Re}(f'(z_0)) & -\operatorname{Im}(f'(z_0)) \\ \operatorname{Im}(f'(z_0)) & \operatorname{Re}(f'(z_0)) \end{pmatrix} .$$

In other words, $f'(z_0) = \left(\frac{\partial u}{\partial x} - \mathbf{i}\frac{\partial u}{\partial y}\right)\Big|_{(x_0,y_0)} = \left(\frac{\partial v}{\partial y} + \mathbf{i}\frac{\partial v}{\partial x}\right)\Big|_{(x_0,y_0)}$. By averaging these two terms, we have

$$f'(z_0) = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathbf{i}\frac{\partial}{\partial y}\right)(u + \mathbf{i}v)\Big|_{(x_0,y_0)} \stackrel{\text{denote}}{=\!=} \frac{\partial}{\partial z}(u + \mathbf{i}v)\Big|_{z_0} .$$

People also like to denote $\frac{\partial}{\partial \bar{z}} :\stackrel{\text{def}}{=\!=} \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathbf{i}\frac{\partial}{\partial y}\right)$. Then the Cauchy-Riemann Equations are equivalent to $\frac{\partial}{\partial \bar{z}}(u + \mathbf{i}v) = 0$.

For example, if $f(x+\mathbf{i}y) = \dfrac{x-\mathbf{i}y}{x^2+y^2}$ defined on $\mathbb{C}\setminus\{0\}$, then easily

$$u_x = \frac{y^2-x^2}{(x^2+y^2)^2} = v_y, \qquad u_y = \frac{-2xy}{(x^2+y^2)^2} = -v_x\,.$$

Thus, assuming that f is complex differentiable (see below), we have

$$f'(x+\mathbf{i}y) = \frac{y^2-x^2+2\mathbf{i}xy}{(x^2+y^2)^2} = \frac{-(x-\mathbf{i}y)^2}{(x^2+y^2)^2} = \frac{-1}{z^2}\,.$$

5.1.3 Verify Analyticity

We also have stated (without proving the "only if") that a function $f = u+\mathbf{i}v$ is complex differentiable on $B(z_0,\delta)$, i.e., analytic at z_0, if and only if the function $(x,y)\mapsto(u,v)$ is of C^1 and satisfies the Cauchy-Riemann Equation on the same neighborhood. This provides us an easy way to check the condition without doing limit.

EXAMPLE 5.1. Let $f(z) = \dfrac{1}{z} = \dfrac{x-\mathbf{i}y}{x^2+y^2}$. Both u and v are clearly of C^1 on $\mathbb{C}\setminus\{0\}$, in particular, on $B(z_0,\delta)$ for any $z_0\neq 0$. We also have checked that the Cauchy-Riemann Equations are valid on $\mathbb{C}\setminus\{0\}$. Thus, we can conclude that f is analytic on $\mathbb{C}\setminus\{0\}$.

EXAMPLE 5.2. We can also use this to get special results. Let g be an analytic function such that $\mathrm{Re}(g)$ is a constant. Then $g(z) = c+\mathbf{i}v(x,y)$. By Cauchy-Riemann Equations,

$$v_x = -u_y \equiv 0, \qquad v_y = u_x \equiv 0 \quad \text{on } \Omega.$$

Since Ω is an open connected set, the only result is that v is also a constant. Therefore, the function g is a constant.

5.2 A Special Function

Let $f(z) = e^x\cos y + \mathbf{i}e^x\sin y$ on $\Omega=\mathbb{C}$. We will first show that f is an entire function. First, both $u = e^x\cos y$ and $v = e^x\sin y$ are of C^∞. Second,

$$u_x = e^x\cos y = v_y, \qquad u_y = -e^x\sin y = -v_x\,; \quad \text{on the whole } \mathbb{C}.$$

Therefore, f is an entire function. Moreover,

$$f'(z) = u_x + \mathbf{i}v_x = e^x\cos y + \mathbf{i}e^x\sin y = f(z)\,.$$

What is the real function that its derivative equals itself on the whole $\mathbb{R}$? Obviously, it is $e^x = f(x+\mathbf{i}0)$ defined above. For this reason, we DEFINE

$$\exp(z) = e^z :\overset{\text{def}}{=\!=} e^x\cos y + \mathbf{i}e^x\sin y\,,$$

and it is called the *exponential function*. Moreover, the expression can be written as

$$e^{x+\mathbf{i}y} = e^x\left(\cos y + \mathbf{i}\sin y\right)\,.$$

When $y=0$, we have $e^{\mathbf{i}y} = \cos y + \mathbf{i}\sin y$ for $y\in\mathbb{R}$. It is obvious that $e^{x+\mathbf{i}y} = e^x e^{\mathbf{i}y}$, which somehow respects the *index law*.