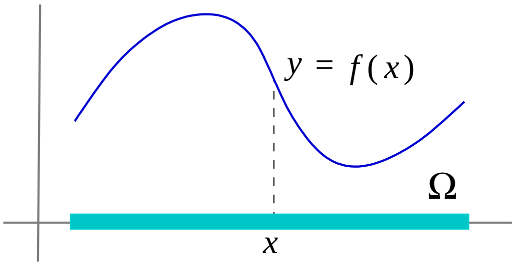
THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2230A (First term, 2015–2016)
Complex Variables and Applications
Notes 2 Complex Functions

2.1 General Principle

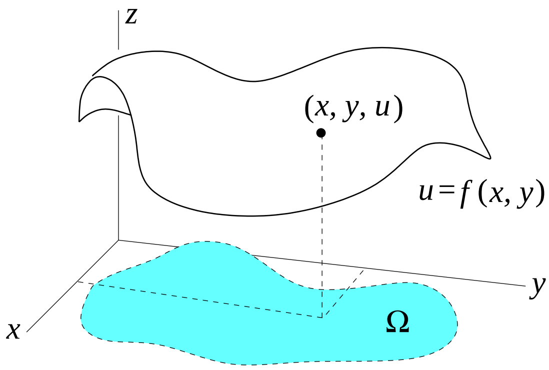
We are studying a continuous function $f: \ \Omega \subset \mathbb{C} \to \mathbb{C}$. So far, the topology (and geometry) of $\mathbb{C}$ is the same as $\mathbb{R}^2$, it is the same as a function $\Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$.

The key question is ***How to*** visualize or understand a function?

For a function $\Omega \subset \mathbb{R} \to \mathbb{R}$, we use the following graph,



For a functiion $\Omega \subset \mathbb{R}^2 \to \mathbb{R}$, we also use the graph.



What about the situation of $f: \ \Omega \subset \mathbb{C} \to \mathbb{C}$?

The general principle is that we take $S \subset \Omega$ and look at its *image*,

$$f(S) \stackrel{\text{def}}{:=\!=} \{\, f(z): \ z \in S \,\} \ .$$

In fact, we do not only look at one single S, but we often look at typical sets S.

On the other hand, we also consider $T \subset \mathbb{C}$ and look at its *pre-image*,

$$f^{-1}(T) \stackrel{\text{def}}{:=\!=} \{\, z \in \Omega: \ f(z) \in T \,\} \ .$$

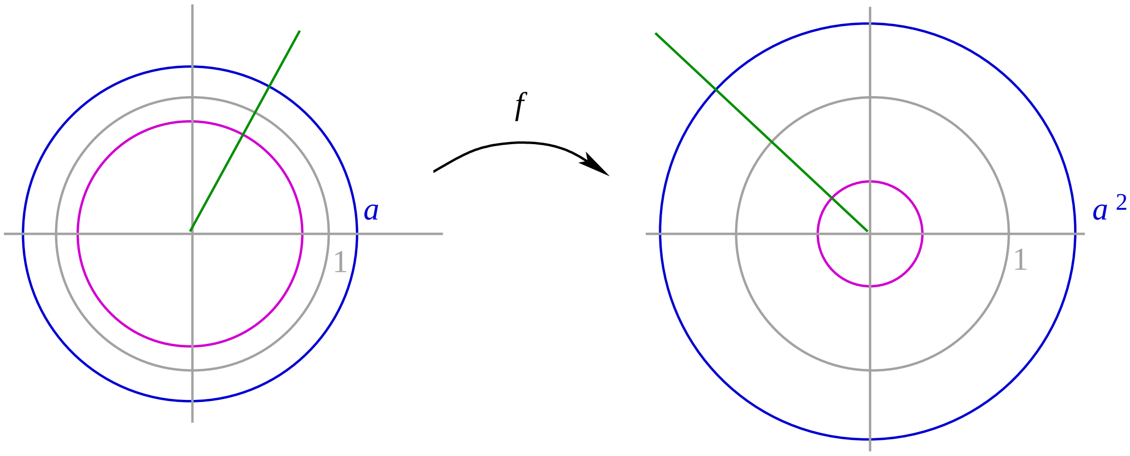
In the following, we will use some examples to illustrate this.

2.2 Example: the Square

Let us consider $f : \Omega = \mathbb{C} \to \mathbb{C}$ where $f(z) = z^2$. In other words, if $z = x + \mathbf{i}y$ and $f(z) = u + \mathbf{i}v$, then $u = x^2 - y^2$ and $v = 2xy$.

2.2.1 Easy Observations

Take a circle S_a with center at the origin and radius $a > 0$, i.e. $S_a = \left\{ ae^{\mathbf{i}\theta} : \theta \in \mathbb{R} \right\}$. Clearly, $f(S_a) = \left\{ a^2 e^{2\mathbf{i}\theta} : \theta \in \mathbb{R} \right\} = \left\{ a^2 e^{\mathbf{i}\phi} : \phi \in \mathbb{R} \right\}$. By taking different radii $a > 0$, we will have the pictures of S_a's and their images.



Similarly, if one takes the radial line $R_\alpha = \left\{ re^{\mathbf{i}\alpha} : r > 0 \right\}$ where $\alpha \in \mathbb{R}$, then its image is also another radial line

$$f(\mathbb{R}_\alpha) = R_{2\alpha} = \left\{ re^{2\mathbf{i}\alpha} : r > 0 \right\} .$$

However, we should be careful that $f^{-1}(R_{2\alpha}) \neq R_\alpha$, in fact,

$$f^{-1}(R_{2\alpha}) = \left\{ re^{\mathbf{i}\alpha} : r > 0 \right\} \cup \left\{ re^{\mathbf{i}(\alpha+\pi)} : r > 0 \right\} = R_\alpha \cup R_{\alpha+\pi} .$$

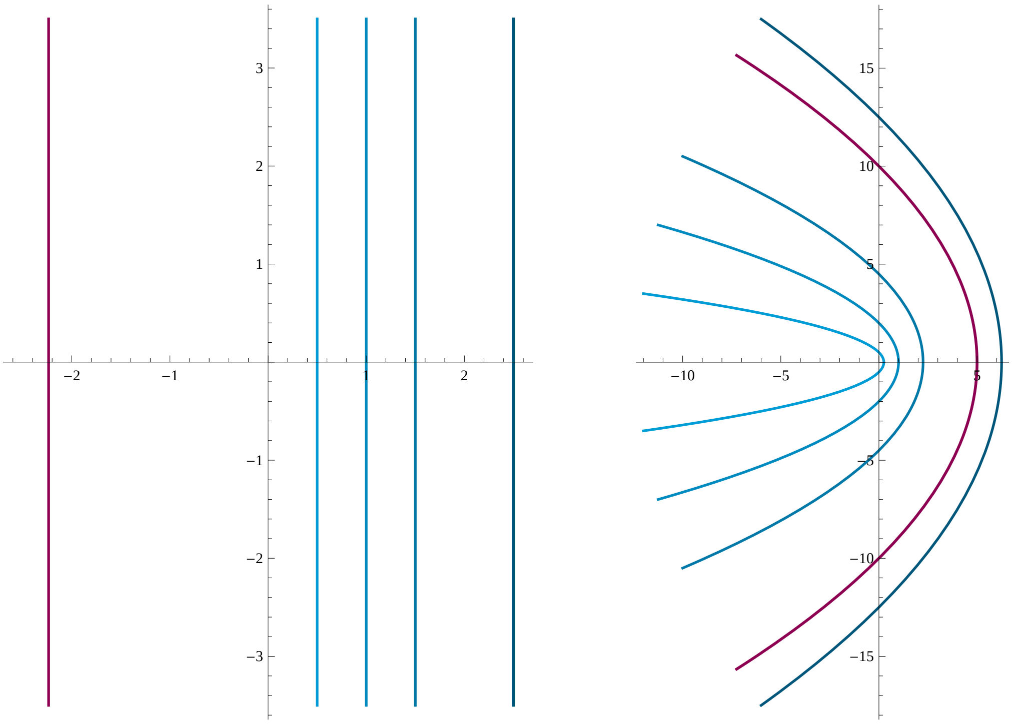
2.2.2 Important Images

Let us start with a vertical line $V_a = \{ a + \mathbf{i}t : t \in \mathbb{R} \}$, $a \neq 0$. ***What*** will its image look like? The essential step is to find out how $f(V_a)$ is mathematically described in the target uv-plane.

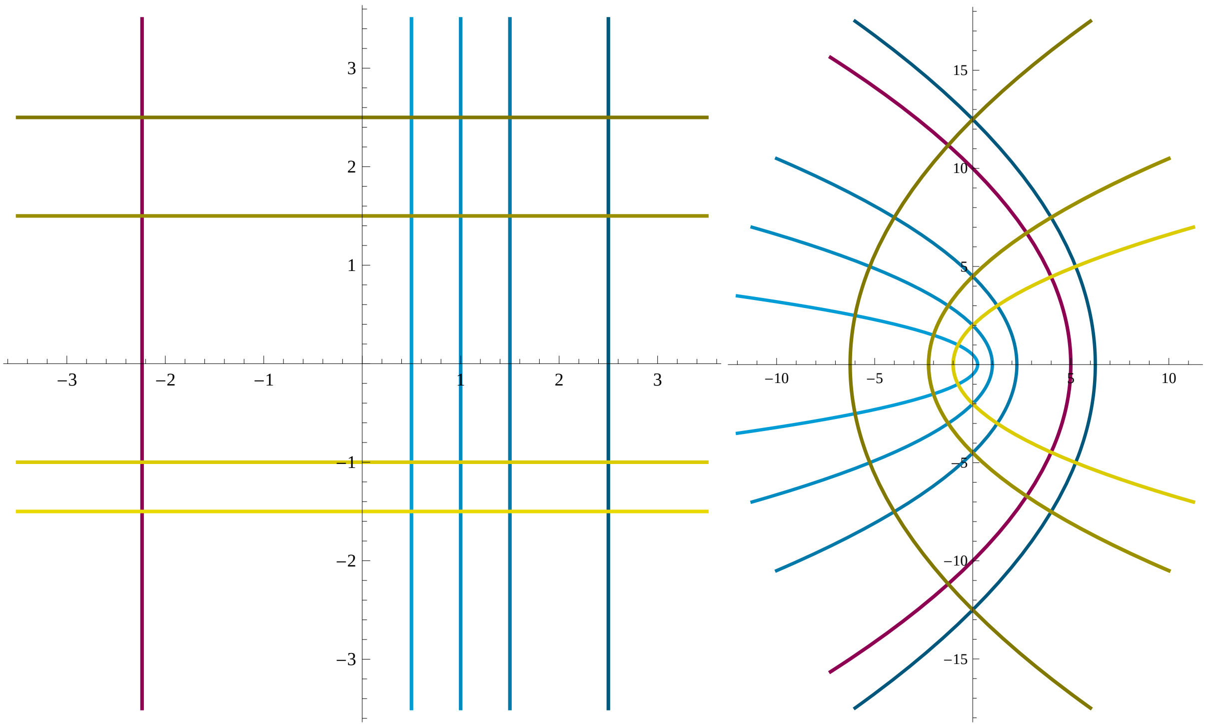
Let $f(a + \mathbf{i}t) = u + \mathbf{i}v$. Then $u = a^2 - t^2$ and $v = 2at$ define a curve with parameter $t \in \mathbb{R}$. By eliminating the parameter t, one shows that it satisfies the equation $u = a^2 - \dfrac{v^2}{4a^2}$, which is a parabola hitting the u-axis at $a^2 > 0$.

Note that this is only one direction of the argument. Mathematically, we have shown $f(V_a)$ is a subset of a parabola. We indeed need to look at the parametric equations to conclude that $f(V_a)$ is the whole parabola and how it is oriented.

In the picture below, taking different values of $a \neq 0$ gives us different parabolas. If $a > 0$, the parabola should be oriented from bottom to top; while for $a < 0$, the parabola from top to bottom. At $a = 0$, the parabola collapses to the negative axis.



If we also consider horizontal lines in $\Omega = \mathbb{C}$, we have another group of parabolas.

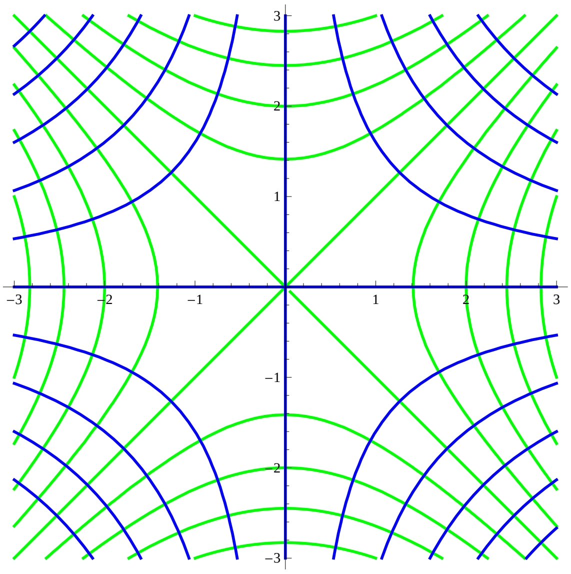


Note that the two groups of parabolas are perpendicular to each other. This is a special property about functions that we will study in this course. It requires the function $f: \ \Omega \subset \mathbb{C} \to \mathbb{C}$ to be ***more than*** continuous. But, this orthogonality fails at the origin, which can be explained later. Many physical situations obey such an orthogonality, for example, heat propagation and isothermal front. This is one reason why complex functions are important in applications.

Similarly, we may consider pre-images of vertical and horizontal lines. Instead of substitution of z into f, this involves an inverse process, i.e., solving equations. For example, in order to find the pre-image of a vertical line, we need to solve

$$x^2 - y^2 = c, \qquad 2xy = t, \qquad \text{where } c \in \mathbb{R} \text{ is fixed and } t \text{ is a parameter.}$$

In general, each straight line will have two "branches" of pre-images. In the picture below, the green ones are pre-images of vertical lines and the blue ones are of horizontal lines.



2.3 Example: Sine function

Let $f:\ \Omega = \mathbb{C} \to \mathbb{C}$ where $f(x + \mathbf{i}y) = \sin(x)\cosh(y) - \mathbf{i}\cos(x)\sinh(y)$. In a future chapter, you will see that this is indeed defined to be $\sin(z)$ for $z \in \mathbb{C}$.

Now, if we write $f(x + \mathbf{i}y) = u + \mathbf{i}v$, where $u(x, y), v(x, y) \in \mathbb{R}$, we have

$$u(x, y) = \sin(x)\cosh(y), \qquad v(x, y) = -\cos(x)\sinh(y)\,.$$

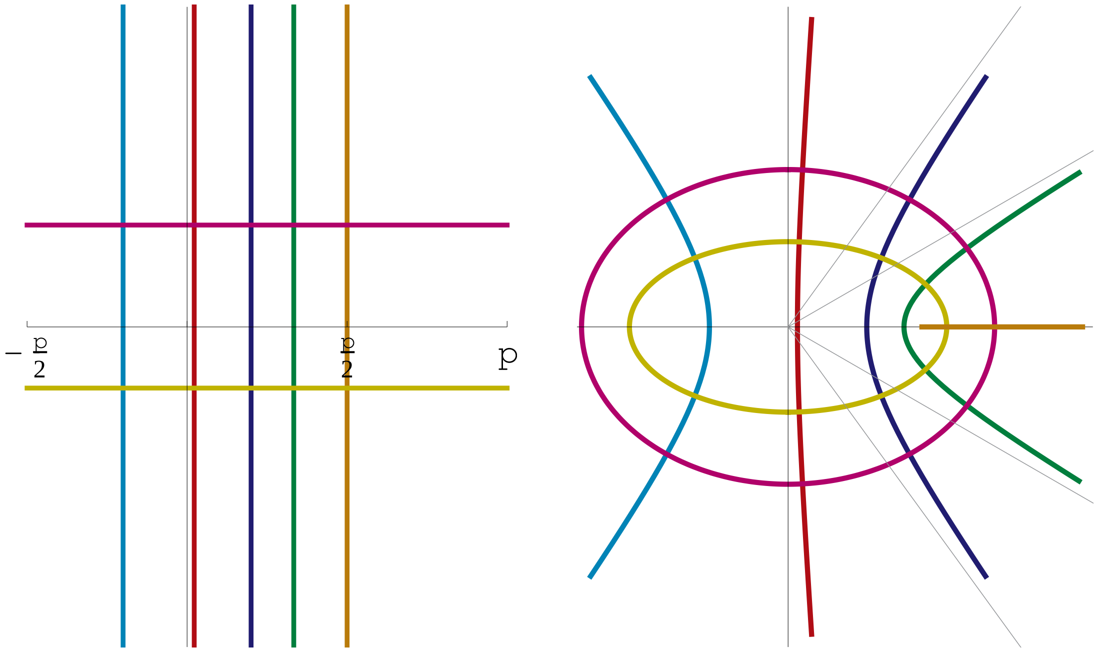
Consider the image of $V_a = \{\, a + \mathbf{i}t :\ t \in \mathbb{R}\,\}$. Then $f(a + \mathbf{i}t) = \sin(a)\cosh(t) - \mathbf{i}\cos(a)\sinh(t)$ gives the parametric form and (u, v) satisfies the equation

$$\frac{u^2}{\sin^2(a)} - \frac{v^2}{\cos^2(a)} = 1, \qquad \text{which is a pair of hyperbola.}$$

Note that the above calculation only shows that $f(V_a)$ is a subset of the hyperbola. In fact, from the parametric form, it should be one branch of the hyperbola.

Starting from $a = 0$, $f(a + \mathbf{i}t) = 0 - \mathbf{i}\sinh(t)$ and the image $f(V_a)$ is only the y-axis. For $0 < a < \pi/2$, $f(V_a)$ is a branch of the hyperbola with orientation going downward because $-\cos(a) < 0$; when $a = \pi/2$, $f(a + \mathbf{i}t) = \cosh(t)$ which only covers the interval $[1, \infty)$. Such phenomenon is shown in the following picture.

In fact, for $\pi/2 < a < \pi$, we still have this picture but the orientation will go upward. Moreover, for $\pi < a < 2\pi$, a similar situation occurs and $f(V_a)$ lies in the left half of the complex plane.



By similar method, one will work out the images of horizontal lines under f, which are ellipses orthogonal to the hyperbolas.

2.4 Example: Möbius transformation

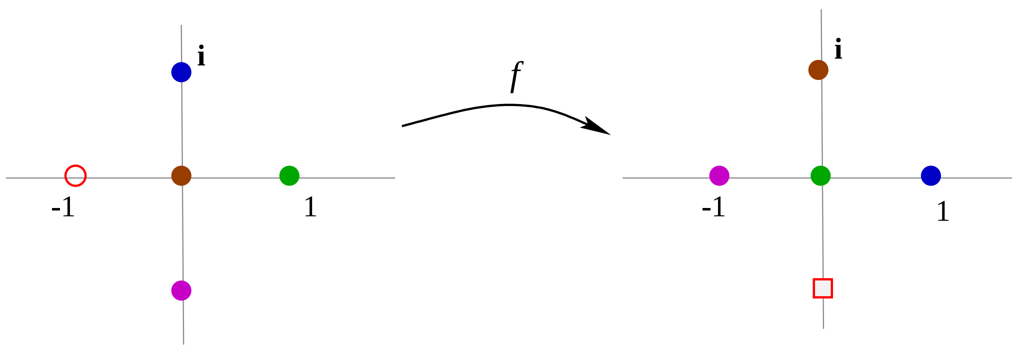
Let us consider $f : \ \Omega = \mathbb{C} \setminus \{-1\} \to \mathbb{C}$ where $f(z) = \dfrac{-\mathbf{i}(z-1)}{z+1}$.

First, it is obvious that f is undefined when $z = -1$, so $\Omega = \mathbb{C} \setminus \{-1\}$. What about the range of f? One may try to solve the equation

$$f(z) = \frac{-\mathbf{i}(z-1)}{z+1} = w, \qquad \text{for an arbitrary } w \in \mathbb{C}.$$

The answer is $z = \dfrac{1+\mathbf{i}w}{1-\mathbf{i}w}$. So, there will be a unique solution for z except $w = -\mathbf{i}$. Thus, $f : \ \Omega \to \mathbb{C} \setminus \{-\mathbf{i}\}$ is a bijection.

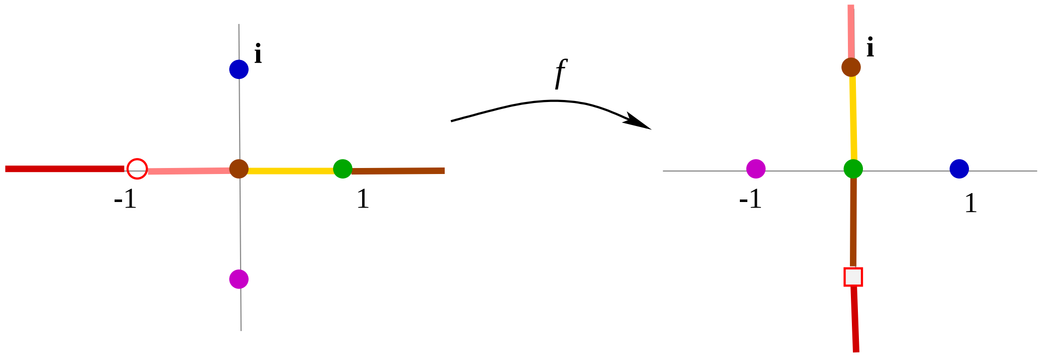
Apparently, it is harder to visualize the action of the function because the images of horizontal or vertical lines are difficult to find. One can always rely on the brute force of plotting point by point (using the computer). Let us first try the action on some typical points, $1, \mathbf{i}, -\mathbf{i}$, which is shown by colors in the picture.



Let us then consider the real axis $\mathbb{R} \setminus \{-1\}$, i.e., $z = x \neq -1$. Clearly, $f(x)$ is purely imaginery, i.e., the image lies in the v-axis. Moreover, we may study the fraction $-(x-1)/(x+1)$ for different ranges of x and get the corresponding images of $f(x)$. For example, if $0 < x < 1$, then $(x-1) < 0$ and $(x+1) > 0$, so the fraction decreases from 1 to 0. As a summary, we have

x	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
$\dfrac{-(x-1)}{x+1}$	$(-\infty, -1)$	$(1, \infty)$	$(0, 1)$	$(-1, 0)$

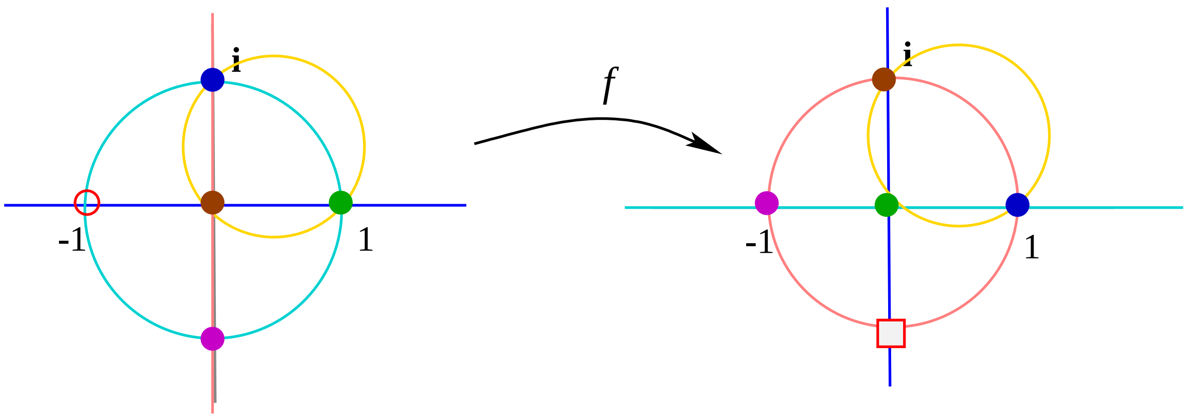
Thus, the image of the x-axis is found in the v-axis and shown below.



Next, we may try to find the image of the y-axis. So, one may calculate $f(\mathbf{i}y) = u + \mathbf{i}v$. Then, it can be checked that $u^2 + v^2 = 1$. That is, the image of the y-axis lies in the unit circle. More details can be observed by showing (as an exercise) that

$$\mathrm{Im}\,(f(z)) = 0 \quad \text{if and only if } |z| = 1.$$

In other words, f sends the unit circle (except $z = -1$) to the real axis. Moreover, by changing the condition to $|z| < 1$ or > 1, you may see the image of the inside and outside of the circle is respectively the upper and lower half-plane. These results are presented by the following picture



This example belongs to a more general class of examples called *Möbius transformation* or *linear fractional transformation*. Such mappings takes (see p. 311–329 of Chapter 8 of the textbook)

$$\{\,\text{circles}\,\} \cup \{\,\text{st. lines}\,\} \longrightarrow \{\,\text{circles}\,\} \cup \{\,\text{st. lines}\,\}\,. \tag{$*$}$$

Since three points determine a circle or straight line, one can always easily find the image and pre-image of the mapping. For example, $-1, 0, 1 \in \Omega$ determine a straight line (the real axis), knowing that $f(-1) = \infty$, $f(0) = \mathbf{i}$, and $f(1) = 0$, which determine the imaginary axis. One immediately concludes that the image of the real axis is the imaginary axis.

2.4.1 General Möbius transformation

In general, a Möbius transformation on $\mathbb{C}$ is given by the form $z \mapsto \dfrac{az+b}{cz+d}$, $\quad a, b, c, d \in \mathbb{C}$.
It is defined on $\Omega = \mathbb{C} \setminus \{-d/c\}$ and its range is $\mathbb{C} \setminus \{a/c\}$. However, we usually formally define that (which is consistent with the sense of taking limit)

$$\frac{-d}{c} \mapsto \infty \quad \text{and} \quad \infty \mapsto \frac{a}{c}\,.$$

Also in Chapter 8 of the textbook, one sees that a Möbius transformation also preserves the angle at the intersection of two circles, or of a circle with a straight line (indeed at the intersection of any two curves). For example, in the above example, the unit circle, the real axis, the imaginary axis are pairwisely orthogonal. Then the same situation occurs to their images, which are the real axis, imaginery axis, and the unit circle. Knowing the properties of $(*)$ above and angle preserving, one may easily obtain the images of many sets in Ω.

In the previous example, let V be a vertical straight line. We first know that $f(V)$ is either a straight line or circle. If V is the vertical line at $z = -1$, then $f(V)$ contains $f(-1) = \infty$ so $f(V)$ must also be a straight line. Moreover, V is a straight line, so we may consider $\infty \in V$, then $f(V)$ contains $f(\infty) = -\mathbf{i}$. That is, $f(V)$ is a straight line passing through $-\mathbf{i}$. Finally, V is vertical so it intersection the real axis at right angle, then $f(V)$ intersects $f(x\text{-axis}) = v\text{-axis}$ at right angle. Hence, $f(V)$ is a horizontal line passing through $-\mathbf{i}$. Similarly, if $-1 \notin V$, then $f(V)$ is a circle passing through $\mathbf{i}$ and its center lies on the imaginary axis.

2.4.2 Other interesting cases

Another special example of Möbius transformation is $z \mapsto \dfrac{1}{z}$. You are encouraged to understand this mapping by looking at its action on typical subsets.

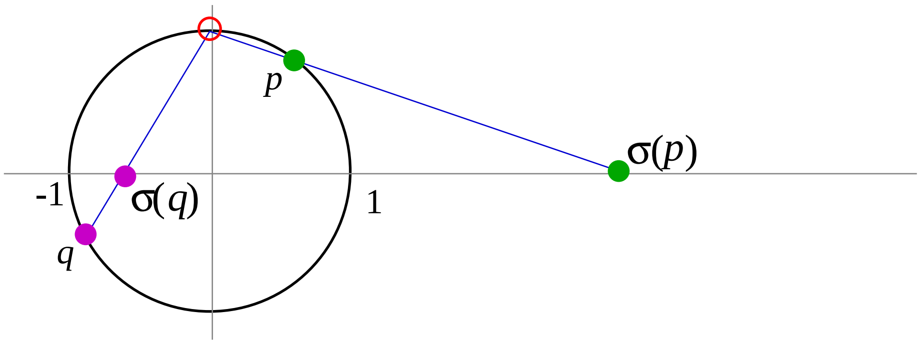
Finally, a related mapping is $\rho(z) = 1/\overline{z}$. Its action is very similar to that of $1/z$. Now, observe that the above example f is taking the unit circle and its inside to the real axis and the upper-half plane. Show that $f \circ \rho \circ f^{-1}(z) = \overline{z}$, i.e., the reflection along the real axis. For this reason, ρ is considered the reflection along the unit circle. All these will be understood from another viewpoint later.

2.5 Stereographic Projection

Complex functions also play an important role in geometry. Part of the reasons is revealed by the above intriguing fact of putting circles and straight lines as a group. For this, we first introduce a bijection between a circular arc and the real line. Mathematically, we denote

$$\mathbb{S}^{n-1} \stackrel{\text{def}}{:=} \{\, \vec{\boldsymbol{x}} \in \mathbb{R}^n : \ \|\vec{\boldsymbol{x}}\| = 1 \,\}\ .$$

Thus, $\mathbb{S}^1$ is the unit circle (center at origin with radius one) and $\mathbb{S}^2$ is the sphere.



The mapping $\sigma : \mathbb{S}^1 \setminus \{(0,1)\} \to \mathbb{R}$ is defined by the above picture. Mathematically, the expression of σ can be easily found by elementary mathematics, e.g., similar triangles. The expression for its inverse is useful,

$$\sigma^{-1}(x) = \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1} \right) .$$

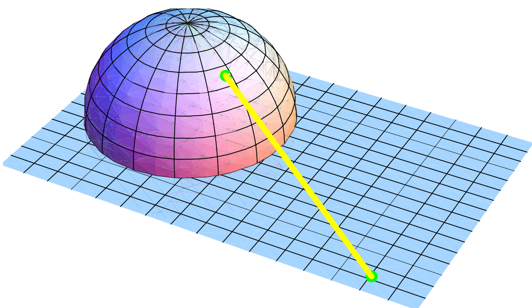
By revolving the above picture about the y-axis, we have a bijection σ from $\mathbb{S}^2 \setminus \{(0,0,1)\}$ to $\mathbb{R}^2 = \mathbb{C}$. Either this mapping σ or its inverse is called the *stereographic projection*. The expression of $\sigma^{-1} : \mathbb{C} \to \mathbb{S}^2$ is given by

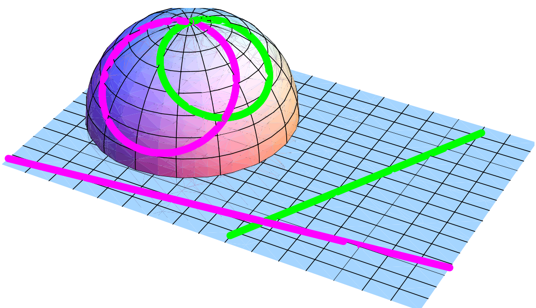
$$\sigma^{-1}(z) = \left(\frac{2\operatorname{Re}(z)}{|z|^2+1}, \frac{2\operatorname{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right) .$$

We have the following correspondence

$$(\text{northern hemisphere, equator, southern hemisphere}) \xrightarrow{\sigma} (\text{outside, unit circle, inside}) .$$

Naturally, we would see the sphere $\mathbb{S}^2$ as $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, called the *Riemann Sphere*.





Straight lines on $\mathbb{C}$ will be corresponding to circles passing through the north pole in $\mathbb{S}^2$. This can be illustrated by the second picture. In fact, the projection rays joining the north pole to the straight line on $\mathbb{C}$ will form a plane cutting the sphere at the desired circle. For circles in $\mathbb{C}$, one may also show that they become other circles in $\mathbb{S}^2$.

Previously, we have discussed the mapping $f : z \mapsto -\mathbf{i}(z-1)/(z+1)$, which maps the unit circle and its inside to the real axis and the upper-half plane. With the stereographic projection, the unit circle and its inside corresponds to the equator and the southern hemisphere; also the real axis and half plane also corresponds to a hemisphere. With this perspective, the mapping f is in fact a rotation on the sphere taking the north-south hemisphere to the east-west hemisphere. Likewise, the mapping $z \mapsto 1/z$ is also a rotation of the sphere taking north-south to south-north.