## math2055 exercise 2

**contents: $\epsilon - N$ definition, $\epsilon - \delta$ definition of limits, intermediate value theorem, extreme value theorem, Bolzano-Weierstrass Theorem**

### Notations

- $\mathbb{N}' = \{1, 2, 3, \cdots\}$
- $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$
- '$\overset{\text{def}}{=}$' will be abbreviated by ':='.

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1. Let $x : \mathbb{N}' \to \mathbb{R}$ ([1]) be given by $x(n) := \dfrac{n^2+n+1}{n^3}$. Show, using the $\epsilon - N$ definition of limit that
$$\lim_{n\to\infty} x(n) = 0$$

2. Consider the sequence
$$x_n := \frac{2^n + 3}{2^n + n + 10}$$
show using the $\epsilon - N$ definition, that $\lim_{n\to\infty} x_n = 1$.

3. In the following question and the next one, we describe two methods to solve $\lim_{x\to 3} x^2 = 9$ using the $\epsilon - \delta$ method.

Method (I)

(1) $|x^2 - 9| < \epsilon$ (Goal)

Want to find $\delta_1$ such that (1) will be satisfied, provided that the following inequality hold:

(2) $0 < |x - 3| < \delta_1$ (Starting Point)

(3) From (1) we obtain that if
$$3 - \underbrace{(3 - \sqrt{9-\epsilon})}_{\delta^*} < x < 3 + \underbrace{(\sqrt{9+\epsilon} - 3)}_{\delta^{**}} \tag{1.1}$$
then $x^2$ satisfies $9 - \epsilon < x^2 < 9 + \epsilon$.

Now if $0 < |x - 3| < \underbrace{\min\{\delta^*, \delta^{**}\}}_{\delta_1}$, then

(4) $\cdots < x < \cdots$
$\cdots < x^2 < \cdots$

(5) From (4), it follows that $|x^2 - 9| < \epsilon$

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[1] This kind of functions are usually called 'sequence' and $x(n)$ is denoted by the symbol $x_n$. The whole sequence is usually denoted by the symbol $\{x_n\}$.

Answer the following question concerning Method (I):

(a) Explain the geometric meaning of the two quantities $\delta^*$ and $\delta^{**}$ in (1.1). (Hint: Drawing a picture may help! They are distance between the point $x = 3$ and some other points (but which points?))

(b) Fill in the missing details in (4)

(c) Show that $\delta^{**} < \delta^*$

> Method (II)
>
> (A) From (1) in Method (I) in the preceding question, we obtain
>
> $$|x-3||x+3| < \epsilon$$
>
> (B) Suppose
>
> $$0 < |x-3| < \delta_2$$
>
> (C) By choosing $\delta_2 = 1$, we obtain from (B)
>
> $$|x+3| \leq 7$$
>
> (D) $|x+3| \leq 7$
>
> (E) Using (B) and (D), we get
>
> $$|x^2 - 3^2| \leq 7\delta_2$$
>
> (F) Given any $\epsilon > 0$, we can choose $\delta_2 :=$ $\boxed{??}$ to complete the proof.

Answer the following questions:

(a) Explain why in (C), we claimed that:

'we obtain from (B) that $|x+3| \leq 7$' ?

(b) What is/are the possible choices of $\delta_2$ in $\boxed{??}$ of Method (II), (F)?

(c) Comparing Method (I) and Method (II), find all values of $\epsilon$ (if any) for which

$$\delta_1 < \delta_2.$$

4. (Exercise on the Extreme Value Theorem)

(a) Let $f : [1,3] \to \mathbb{R}$ be a function. Write down the definition of

$\boxed{f \text{ is not differentiable at } x = 2.}$

(b) Let $c$ be a point in the set $[1,3]$ in part (a). Write down your definition of

$\boxed{c \text{ is an } \underline{\text{absolute maximum point}} \text{ of } f.}$

(c) In the preceding definition, state whether it is necessary to assume that the function is continuous. Explain why you think it is necessary/not necessary.

(d) Define a function $g : [1, 3] \to \mathbb{R}$ such that (i) $g$ is continuous at every point in $[1, 3]$, (ii) $g$ is not differentiable at the point $x = 2$, (iii) $f$ has an absolute maximum point at $x = 1$ and (iv) an absolute minimum point at $x = 2$.

5. (Intermediate Value Theorem) Using the intermediate value theorem, show that the equation

$$x^3 - \sin 100x - 1000 = 0$$

has a solution in the domain $[0, N]$ for some sufficiently large natural number $N$.

6. (Bolzano-Weierstrass Theorem) Consider the sequence $\{x_n\}$ defined by

$$x_n := (-1)^n \left( \frac{2^n - 1}{2^n + 1} \right)$$

(a) Show, using school mathematics, that it is bounded above and bounded below.

(b) Find two subsequences of $\{x_n\}$ (give them the names $\{x_{n'_k}\}$ and $\{x_{n''_k}\}$) which are convergent.