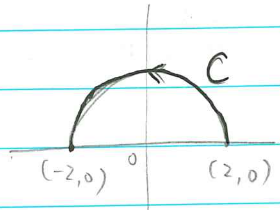
MATH2550

Course Work 3

1. Find $\int_C y\, ds$ where C is the upper semi-circle of radius 2, centered at $(0,0)$ and running from $(2,0)$ to $(-2,0)$.
2. Repeat question 1 with the curve C replaced by a quarter circle of radius 2 running from $(2,0)$ to $(0,2)$.
3. Repeat question 2 with the curve C now being the line segment joining $(2,0)$ to $(0,2)$.
4. Write down the equation of the tangent line to the curve $y = 1/x$ at any point $(a, 1/a)$. Here a is assumed to be a positive number. Then, find (in terms of a) the 2 points where the tangent line intersects the x −axis, respectively the y −axis. Finally, show that the areas the right-angle triangles formed by such tangent lines & the two axis is always the same.
5. Write down equation of tangent plane to the surface $z = xy$ at the point $x = 1, y = -1$. Find also a unit normal vector to this plane.

1.



① Parametrize C : $\alpha : [0, \pi] \longrightarrow \mathbb{R}^2$ defined by

$$\alpha(t) = (2\cos t,\ 2\sin t)$$

i.e. $\begin{cases} x(t) = 2\cos t \\ y(t) = 2\sin t \end{cases}$, $t \in [0, \pi]$

② $f(x,y) = y$,

express f as a function of the new parameter t.

$$f(x(t), y(t)) = y(t) = 2\sin t$$

③ Find the "speed" of the curve C.

i.e. Compute $|\alpha'(t)|$

$$\alpha'(t) = (x'(t), y'(t)) = (-2\sin t,\ 2\cos t)$$

$$|\alpha'(t)| = \sqrt{|x'(t)|^2 + |y'(t)|^2} = 2$$

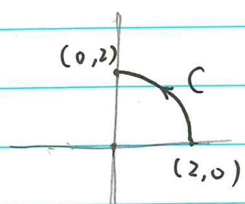
④ Compute $\int_C f\, ds$

$$\int_C f\, ds = \int_0^{\pi} f(x(t), y(t)) \cdot |\alpha'(t)|\, dt$$

$$= \int_0^{\pi} 2\sin t \cdot 2\, dt = 4\int_0^{\pi} \sin t\, dt$$

$$= -4 \cos t \Big|_0^{\pi} = -4(-1-1) = 8$$

2.



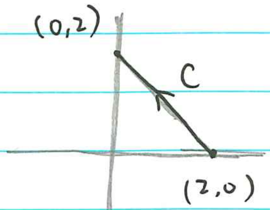
① $\alpha : [0, \frac{\pi}{2}] \longrightarrow \mathbb{R}^2$,

$$\alpha(t) = (2\cos t,\ 2\sin t) = (x(t), y(t))$$

②, ③ same as Q1

④ $\int_C f\, ds = \int_0^{\frac{\pi}{2}} 4\sin t\, dt = -4\cos t \Big|_0^{\frac{\pi}{2}} = -4(0-1) = 4$

3.



① Parametrize C :

$\alpha : [0,1] \longrightarrow \mathbb{R}^2$ defined by

$$\alpha(t) = (x(t), y(t)) = \underbrace{(2,0) + t(-2,2)}_{\text{expression of a straight line}} = (2-2t, 2t)$$

② $f(x,y) = f(x(t), y(t)) = 2t$

③ $\alpha'(t) = (-2, 2)$, $|\alpha'(t)| = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$

④ $\int_C f \, ds = \int_0^1 2t \cdot 2\sqrt{2} \, dt$ $\quad \left(= \int_0^1 f(x(t), y(t)) \cdot |\alpha'(t)| \, dt \right)$

$= 4\sqrt{2} \int_0^1 t \, dt = 2\sqrt{2}$

4. ① ~~Implicit~~ Direct differentiation : (view y as a function of x)

$y = \frac{1}{x}$, $\Rightarrow \frac{dy}{dx} = -\frac{1}{x^2}$

$\left.\frac{dy}{dx}\right|_{x=a} = -\frac{1}{a^2}$ (slope)

So the tangent line :

$$y = -\frac{1}{a^2}(x-a) + \frac{1}{a} = -\frac{1}{a^2}x + \frac{2}{a}$$

② Consider linear approximation (Taylor expansion to the 1^{st} order) :

$f(x,y) = \frac{1}{x} - y = 0$ ······ expression of the curve

$f_x = -\frac{1}{x^2}$, $f_y = -1$

$f_x(a, \frac{1}{a}) = -\frac{1}{a^2}$, $f_y(a, \frac{1}{a}) = -1$

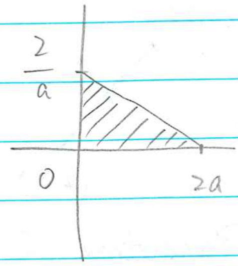
$T_1(x,y) = f(a, \frac{1}{a}) + f_x(a, \frac{1}{a}) \cdot (x-a) + f_y(a, \frac{1}{a}) \cdot (y - \frac{1}{a})$

$= 0 - \frac{1}{a^2}(x-a) - (y - \frac{1}{a})$

$= -\frac{1}{a^2}x - y + \frac{2}{a}$

So the line is : $-\frac{1}{a^2}x - y + \frac{2}{a} = 0$ (Set $T_1(x,y) = 0$)

$y = -\frac{1}{a^2}x + \frac{2}{a}$



y - intercept : $x = 0 \Rightarrow y = \frac{2}{a}$

x - intercept : $y = 0 \Rightarrow x = 2a$.

Hence Area of triangle $= \frac{1}{2} \cdot 2a \cdot \frac{2}{a}$

$= 2$ (It's constant)

5. ① $f(x, y, z) = xy - z = 0$.

$f_x = y, \quad f_y = x, \quad f_z = -1$

$\Rightarrow$ When $x = 1, \ y = -1, \quad z = xy = -1$.

$f_x(1,-1,-1) = -1, \quad f_y(1,-1,-1) = 1, \quad f_z(1,-1,-1) = -1$

$\Rightarrow T_1(x, y, z) = f(1,-1,-1) + (-1)\cdot(x-1) + 1\cdot(y+1) + (-1)\cdot(z+1)$

$= 0 - (x-1) + (y+1) - (z+1)$

For the tangent plane, set $T_1(x, y, z) = 0$.

$\Rightarrow \quad x - y + z - 1 = 0$

② Directly view z as a function of x, y.

$z(1,-1) = -1$

$\frac{\partial z}{\partial x} = y, \quad \frac{\partial z}{\partial x} = x \quad \Rightarrow \frac{\partial z}{\partial x}(1,-1) = -1, \quad \frac{\partial z}{\partial x}(1,-1) = 1$

$z = -1 + (-1)\cdot(x-1) + 1\cdot(y+1)$

$z = -1 - x + 1 + y + 1$

$\Rightarrow \quad x - y + z - 1 = 0$

Normal of the plane : $(1, -1, 1)$ (coefficients of x, y, z).

Reason : for a plane P: $ax + by + cz + d = 0$,

(a, b, c) is a normal vector to P.

let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in P$,

$\begin{cases} ax_1 + by_1 + cz_1 + d = 0 \\ ax_2 + by_2 + cz_2 + d = 0 \end{cases} \Rightarrow a(x_1 - x_2) + b(y_1 - y_2) + c(z_1 - z_2) = 0$.

(i.e. $(a,b,c) \perp$ any lines inside P)