MATH1010-i
Week 5 – 7
(part 2)

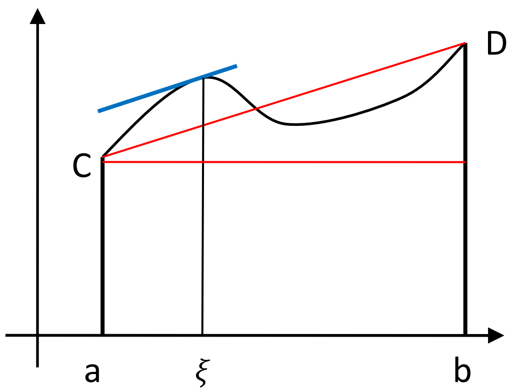
Topics

- RT, LMVT, CMVT, TT
- Applications of Mean Value Theorems
- Taylor's Theorem $n = 1$ version
- Taylor's Theorem $n = 2$ version
- How to do the general case
- Applications of TT

Mean Value Theorems for Derivatives

There are many such theorems, they have all one common theme, i.e. they are about a formula relating the "mean value of a function" to its "derivative" at some unknown point.

Consider the picture



Observation:

- Hypotenuse of the red right-angled triangle is "parallel" to the blue tangent line at $x = \xi$. (note that we are assuming $a < b$).

$$f'(\xi) = \frac{f(b) - f(a)}{b - a} \quad \exists \xi \in (a, b)$$

Theorem

Assumptions:

1. f is differentiable in (a, b).
2. f is continuous on $[a, b]$. (Technical assumption to get an absolute extremum (最大，最小) point)

Conclusion: $\exists \xi \in (a, b)$:

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

A generalization of LMVT is:

Theorem (Cauchy)

Assumptions

1. f, g are both differentiable in (a, b)
2. f, g are both continuous on $[a, b]$.
3. $g'(x) \neq 0 \; \forall x \in (a, b)$. (this condition makes sure that the denominators are non-zero!)

Conclusion: $\exists \xi \in (a, b)$:

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Remark: (LMVT as a special case of CMVT) This is because if we let $g(x) = x$ in CMVT, then $g'(x) = x, \forall x$ inside the domain. Therefore, the equation

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

takes the form $\frac{f'(\xi)}{1} = \frac{f(b)-f(a)}{b-a}$, since $g(b) = b, g(a) = a$ in this case.

As a consequence, we get back LMVT.

Relation between Taylor's Theorem when $n = 1$ and LMVT

If in LMVT, we let $b = x$, and $a = c$, then we obtain the following

$$f'(\xi) = \frac{f(x) - f(c)}{x - c}$$

After multiplying both sides by $x - c$, we get

$$\begin{aligned} f(x) &= f'(\xi)(x - c) + f(c) \\ &= f(c) + f'(\xi)(x - c). \end{aligned}$$

Which is just Taylor's Theorem in the case for the case $n = 1$.

The term $f'(\xi)(x-c)$ is called ther "error term". The reason for this name is due to

$$f'(\xi)(x-c) = \underbrace{f(x)}_{correct\ value\ of\ f\ at\ the\ point\ x} - \underbrace{f(c)}_{approximate\ value\ of\ f}$$

Error term is the difference between the correct value at x and its approximate value at x. (now the "approximate value" is a constant number).

Remark:

The error term depends on $(x-c)^1$, i.e. it is of "degree 1". As $x-c$ increases, the error may increase (of course it also depends on $f'(\xi)$, where ξ depends on x)

Taylor's Theorem for $n = 2$.

We start with asking ourselves the

Question: Can we improve this (i.e make the error smaller, make the approximation on the right-hand side more accurate?)

Answer: Yes. We can try next $f(x) = f(c) + f'(c)(x-c)$+ Error term if we stop at the $(x-c)^1$ (i.e. "1st power of x" term),

Where this time, the "Error" term is of the form

$$A(x-c)^2$$

I.e.

$$f(x) - f(c) - f'(c)(x-c) = A(x-c)^2$$

Goal: Find a formula for the number "*A*".

To see this: Rewrite the above equation as

$$\frac{f(x) - f(c) - f'(c)(x-c)}{(x-c)^2} = K$$

Goal: "Show that $K = \frac{f''(\eta)}{2!}$."

Answer: We think of the left-hand side as a quotient of two functions like what have in "CMVT", More precisely,.let $A(x) = f(x) - f(c) - f'(c)(x-c)$,

$$B(x) = (x-c)^2$$

Note that actually the first line above is just $A(x) - A(c)$, the second line is $B(x) - B(c)$.

So we have $\frac{A(x)-A(c)}{(x)-B(c)} = \frac{A'(\eta)}{B'(\eta)}$ by using CMVT.

Since $A'(\eta) = \left.\frac{dA(x)}{dx}\right|_{x=\eta} = \left.\frac{d\{f(x)-f(c)-f'(c)(x-c)\}}{dx}\right|_{x=\eta} = f'(\eta) - f'(c) \cdot 1$

and $B'(\eta) = \left.\frac{dB(x)}{dx}\right|_{x=\eta} = \left.\frac{d(x-c)^2}{dx}\right|_{x=\eta} = 2(\eta - c)$

This means $\frac{f(x)-f(c)-f'(c)(x-c)}{(x-c)^2} = \frac{f'(\eta)-f'(c)}{2(\eta-c)}$ .... (1)

Now the right-hand side of the above equation is $\left(\frac{1}{2}\right) \cdot \frac{f'(\eta)-f'(c)}{(\eta-c)}$ and the yellow colored part can be calculated using LMVT to get

$$\frac{f'(\eta) - f'(c)}{(\eta - c)} = f''(\beta)$$

for some number β between η and c.

Now putting this back to the right-hand side of (1), we obtain

$$\frac{f(x) - f(c) - f'(c)(x-c)}{(x-c)^2} = \frac{f'(\eta) - f'(c)}{2(\eta - c)} = \left(\frac{1}{2}\right) f''(\beta)$$

which means $\frac{f(x)-f(c)-f'(c)(x-c)}{(x-c)^2} = \left(\frac{1}{2}\right) f''(\beta)$ i.e.

$$f(x) - f(c) - f'(c)(x-c) = \left(\frac{1}{2!}\right) f''(\beta)$$

or

$$f(x) = f(c) + f'(c)(x-c) + \left(\frac{1}{2!}\right) f''(\beta)$$

which is T.T. for $n = 2$.

Summary

We have proved T.T. for the cases $n = 1$ and $n = 2$. Using similar idea, one can prove T.T. for $n = 3,4,\cdots$ What one needs to do is to (i) apply CMVT many times and (ii) apply LMVT once.

In short, Taylor Theorem is nothing but one of the mean value theorems.

1st application of the $n = 2$ Taylor's Theorem – eqn. of tg. line at $x = c$.

As we know, the $n = 2$ T.T. says

$$f(x) = f(c) + f'(c)(x - c) + \left(\frac{1}{2!}\right) f''(\beta)(x - c)^2$$

where $\left(\frac{1}{2!}\right) f''(\beta)(x - c)^2$ is the error term.

Question: What is the meaning of the term $f(c) + f'(c)(x - c)$ on the right-hand side?

Answer: If we put a "subject" y and write $y = f(c) + f'(c)(x - c)$, this is just the equation of the tangent line to the curve $y = f(x)$ at the point $x = c$.

Example
Find the equation of the tangent line to $y = e^x$ at $x = 1$.
Answer: $y = e^1 + e^1(x - 1) = e + e(x - 1)$.

Comment:
We sometimes say $y = f(c) + f'(c)(x - c)$ is the first order (一阶) approximation of the function f near the point c.

Another Application of the Mean Value Theorems – L'Hôpital Rule (in the future, we'll just write L'H Rule).

We just prove the simplest case of L'H Rule, i.e. the case where we assume:

f, g are two differentiable functions in the domain (a, b),
f, g are both continuous functions on the domain $[a, b]$.
For some point $c \in (a, b)$, it holds that $f(c) = g(c) = 0$.
We have limit of the form $\lim_{x \to c} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$.

Then we can conclude that $\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$, provided the limit of $\frac{f'(x)}{g'(x)}$ exists.

Proof idea (as application of CMVT)

The proof is based on the following:

- Interpret $\frac{f(x)}{g(x)}$ as $\frac{f(x)-f(c)}{g(x)-g(c)}$.
- Apply CMVT to this and get $\frac{f(x)-f(c)}{g(x)-g(c)}=\frac{f'(\xi)}{g'(\xi)}$ for some ξ between x and c (two cases here: (i) $x<c$ and (ii) $x>c$. In both cases ξ is sandwiched by x and c, so as $x\to c$, it follows that $\xi\to c$. One can also write this in the form $\lim_{x\to c}\xi=c$.)
- **(Taking limit $\boldsymbol{x\to c}$)** We take this limit for the equation $\frac{f(x)-f(c)}{g(x)-g(c)}=\frac{f'(\xi)}{g'(\xi)}$ and get $\lim_{x\to c}\frac{f(x)-f(c)}{g(x)-g(c)}=\lim_{\xi\to c}\frac{f'(\xi)}{g'(\xi)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$
- The last step (i.e. the yellow-colored one) is obtained by changing the name ξ back to x.

An Example of L'H Rule

Find the limit $\lim_{x\to 0^+}x^x$ (though we have not proved it, the L'H Rule holds also for one-sided limits).

Answer: Write $x^x=e^{x\ln x}$

Then all we need to do is to compute the limit $\lim_{x\to 0^+}x\ln x$.

This one can be computed by considering $\frac{\ln x}{1/x}$, which when $x\to 0^+$ is a limit of ∞/∞ form (L'H Rule also holds in this case, though we also haven't proved it).

Using L'H Rule, we now get $\lim_{x\to 0^+}\frac{\ln x}{1/x}=\lim_{x\to 0^+}\frac{\frac{d\ln x}{dx}}{\frac{dx^{-1}}{dx}}=\lim_{x\to 0^+}\frac{x^{-1}}{-x^{-2}}=0^-$

Using this, we get $\lim_{x\to 0^+}x^x=\lim_{x\to 0^+}e^{x\ln x}=e^{\lim_{x\to 0^+}x\ln x}=1^-$.

Exercise for you. Find $\lim_{x\to 0^+}x^{\sin x}$