MATH1010 I

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Webpage: (generic) www.math.cuhk.edu.hk/course/1819/math1010

Introduction. Calculus is mainly a study of functions.

A function is a "rule" assigning to each "input" x one and only one (a.k.a. "unique") output y.

Diagrammatically we can represent the above line by $x \mapsto y$.

Remark. There is a "vertical bar" before the arrow!

Later on, we will use an "arrow-without-vertical-bar" to mean something else!

Remark. Since y is the "result of applying (a rule) to" x, we can remember this by letting $y = f(x)$ (the right-hand side means "applying the rule f to the input x).

Terminology (= name) We give the name "domain" to the set from which we pick our inputs such as x .

Example. (of "function", "domain")

Consider the rule $x \mapsto \underbrace{x^2}_{y}$

In plain English, the rule says: whenever we input x the output is x squared.

Usually in textbooks, you just see the rule in the form of an equation like

$$y = \text{result of the rule applied to } x$$

More concise way to say is like

$$y = f(x)$$

where f denotes the "rule".

Remark. In the above, we talked about the word "rule" in simple ways using English words. We can also talk about "rule" using Set Theory via something called Cartesian Products. But we won't do this here.

Example. The above function can be described by $y = x^2$

or if you like $y = f(x) = x^2$ (the first "=" signs means "y is the 'result' of applying the (rule) f to the input x". The second "=" sign means the rule f applied to the input x is x^2).

In this example, the domain is $(-\infty, +\infty)$.

Of course, you can choose a smaller set as its domain, e.g. $(0, \infty)$ or $[0, \infty)$.

Notation. $(0, \infty) = \{x \text{ is a real no.} \mid \ 0 < x < \infty\}$

$$[0, \infty) = \{x \text{ is a real no.} \mid \ 0 \leq x < \infty\}$$

Sequence. If now you choose the domain to be the set of all natural numbers, i.e. the set (denoted by the symbol) $\mathbb{N}$ and given by the formula

$$\mathbb{N} = \{1,2,3,\cdots\}$$

to the function we have been considering, we obtain

$$x \mapsto x^2$$

where now x is a natural number.

To emphasize the fact that x is now a natural number, we use the letter $n, m, k, l, \cdots$ instead of w, x, y, z and write

$$n \mapsto n^2$$

or $y = a(n) = n^2$.

Remark. I changed the "symbol" for the rule from "f" to "a".

Important!!! Such functions whose "domain" is the set of natural numbers, are called sequences and are traditionally denoted by "subscripts", i.e.

$$y = a_n = n^2$$

Remark. We will start the natural numbers from 1 onward. Some other people may like to start it from 0.

Let's give the notation $\mathbb{N}_0$ to mean the "set of all natural numbers including 0".

Example. Consider the sequence given by $a_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$

If you use summation notation, the above line becomes $a_n = \sum_{k=0}^{k=n-1} \frac{1}{2^k}$

Meaning. When $n = 1$, this gives $\sum_{k=0}^{k=1-1} \frac{1}{2^k} = \frac{1}{2^0}$

When $n = 2$, this gives $\sum_{k=0}^{k=2-1} \frac{1}{2^k} = \frac{1}{2^0} + \frac{1}{2^1}$

Then we can show that $\lim_{n\to\infty} a_n = 2$.

Method. We use "long division" to get for any given number b , the following formula

$\frac{b^n-1}{b-1} = 1 + b + b^2 + \cdots + b^{n-1}$

(You can verify this also by "multiplying both sides" by $b - 1$).

Having done this, we choose our b to be given by $b = \frac{1}{2}$. Then we get

$$\frac{\left(\frac{1}{2}\right)^n - 1}{\left(\frac{1}{2}\right) - 1} = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^{n-1} = a_n$$

The leftmost term and the rightmost term give

$$\frac{\left(\frac{1}{2}\right)^n - 1}{\left(\frac{1}{2}\right) - 1} = a_n$$

So we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\left(\frac{1}{2}\right)^n - 1}{\left(\frac{1}{2}\right) - 1} = \frac{\lim_{n\to\infty}\left(\frac{1}{2}\right)^n - \lim_{n\to\infty} 1}{\lim_{n\to\infty}\left(\frac{1}{2}\right) - \lim_{n\to\infty} 1} = -\frac{1}{\left(\frac{1}{2}\right) - 1} = 2.$

Reason. Since the number $\frac{1}{2}$ is between 0 and 1, as n becomes larger and larger, the expression $\left(\frac{1}{2}\right)^n$ goes nearer and nearer to 0.

Remark. Actually this example is a simple case of a general principle to find limit of a sequence.

The principle says:

Assumption: Suppose the sequence a_n satisfies

(i) a_n is bounded from above, i.e. there exists some number M which is bigger than or equal to each and every term of the sequence. Put mathematically, it says: "There exists some M such that $a_n \le M$ for each natural number n.

(ii) $a_n \le a_{n+1}$ for each natural number n.

Conclusion. $\lim_{n\to\infty} a_n$ exists (and is less than or equal to M).

Remark. The second condition is called "increasing". If the inequality is strict, i.e. $<$, then we say the sequence is "strictly increasing".

Terminology. A concise way to describe this principle is:

"an increasing sequence bounded from above has a limit".

Similarly, we have

"a decreasing sequence bounded from below has a limit".

Important Point.

Note that the "upper bound" is not unique. E.g. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n-1}}+\cdots$ is bounded above by any of the following numbers:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n-1}}+\cdots \leq 2$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n-1}}+\cdots \leq 2.01$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n-1}}+\cdots \leq 2.02$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n-1}}+\cdots \leq 2+any\ positive\ no.$$

As a consequence, any number of the form $2+pos.no.$ is an "upper bound" of the sequence $\{a_n\}$ given by

$$a_n = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n-1}}$$

The $+,-,\times,\div$ of Limits of Sequences

We have (under the assumption that (*) $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$ and both A, B are finite numbers), we have

(1) $\lim_{n\to\infty} c \cdot a_n = c \cdot \lim_{n\to\infty} a_n$

(2) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

(3) $\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

(4) $\lim_{n\to\infty} (a_n / b_n) = \lim_{n\to\infty} a_n \,/\, \lim_{n\to\infty} b_n$

For (4), we need to assume that $b_n \neq 0, B \neq 0$.

Remark. Sometimes, we can relax the assumptions and allow the limit to be infinity.

Three Methods to ensure limit of a sequence exists.

(1) an increasing/decreasing sequence bounded from above/below has a limit;

(2) (Sandwich/Squeeze Theorem) Suppose we have 3 sequences $\{b_n\}, \{a_n\}, \{c_n\}$ satisfying $b_n \le a_n \le c_n$. Furthermore, suppose that $\lim_{n\to\infty} b_n = L$ and $\lim_{n\to\infty} c = L$, then the sequence a_n "sandwiched between" b_n and c_n also have limit, i.e. $\lim_{n\to\infty} c_n = L$.

(3) (Sequence going to zero $\times$ bounded sequence has limit going to zero) Mathematically, this is written as:

Suppose $\{a_n\}$ satisfies $\lim_{n\to\infty} a_n = 0$ (i.e. it goes to zero)

and $\{b_n\}$ satisfies $A \le b_n \le B$ (i.e. it is bounded between the numbers A and B),

then the "new" sequence $\{a_n b_n\}$ satisfies

$$\lim_{n\to\infty} a_n b_n = 0.$$

Example. (for (2))

Consider the sequence $\{a_n\}$ defined by $a_n = \sum_{k=1}^{n} \frac{k}{2n^2+k}$.

Then it satisfies the inequalities

$$\sum_{k=1}^{n} \frac{k}{2n^2+n} < a_n < \sum_{k=1}^{n} \frac{k}{2n^2}$$

(Reason: the larger the denominator, the smaller the fraction)

Now we consider the sequence $b_n = \sum_{k=1}^{n} \frac{k}{2n^2+n} = \frac{1}{2n^2+n} \sum_{k=1}^{n} k = \frac{1}{2n^2+n} \cdot (n+1) \cdot \frac{n}{2}$

as well as the sequence $c_n = \sum_{k=1}^{n} \frac{k}{2n^2+n} = \frac{1}{2n^2} \sum_{k=1}^{n} k = \frac{1}{2n^2} \cdot (n+1) \cdot \frac{n}{2}$

Since $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2} \cdot \frac{n(n+1)}{2n^2+n} = 1/4$, $\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{n(n+1)}{4n^2} = 1/4$

We can conclude that $\lim_{n\to\infty} a_n = 1/4$.

Example. (for (3))

Consider the sequences $\{a_n\}$ given by $a_n = 1/n$ and $\{b_n\}$ given by $b_n = \sin(n)$, then we have (i) $\lim_{n\to\infty} a_n = 0$ & (ii) $-1 \le \sin(n) \le 1$ (i.e. $\sin(n)$ is bounded between -1 and $+1$). Hence $\lim_{n\to\infty} \left(\frac{1}{n}\right) \sin(n) = 0$.

Newton's Method for Finding Square Root of a Number

We omit this for the time being, since some of the students have no Calculus background. We will come back to this example later.