THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT 5000 Analysis I 2015-2016

Suggested Solution to Problem Set 7

1. Let $\varepsilon > 0$ be given. Take $\delta = \varepsilon$, then for any tagged partition $x_0, \cdots, x_n$ and $t_0, \cdots, t_{n-1}$ with mesh $< \delta$, we have
$$\begin{aligned}\left|\sum_{i=0}^{n-1} f(t_i)(x_{i+1}-x_i) - 1\right| &= \left|f(t_0)(x_1-x_0) + \sum_{i=1}^{n-1}(x_{i+1}-x_i) - \sum_{i=0}^{n-1}(x_{i+1}-x_i)\right| \\ &= |(f(t_0)-1)(x_1-x_0)| \\ &< \varepsilon.\end{aligned}$$
Since $\varepsilon > 0$ is arbitrary, f is Riemann integrable on $[0,1]$ and $\int_0^1 f = 1$.

To see that f is Darboux integrable, note that
$$U_{f,P} = \sum_{i=0}^{n-1}(x_{i+1}-x_i) \sup_{x\in[x_i,x_{i+1}]} f(x) = 1$$
and
$$L_{f,P} = \sum_{i=0}^{n-1}(x_{i+1}-x_i) \inf_{x\in[x_i,x_{i+1}]} f(x) = \sum_{i=1}^{n-1}(x_{i+1}-x_i) = 1-(x_1-x_0).$$
Hence,
$$U_f = \overline{\int_0^1} f = 1 = \underline{\int_0^1} f = L_f$$
and f is Darboux integrable with $\int_0^1 f = 1$.

2. Let $\varepsilon > 0$ be given. Take $\delta = \varepsilon$, then for any tagged partition $x_0, \cdots, x_n$ and $t_0, \cdots, t_{n-1}$ with mesh $< \delta$, we have
$$\begin{aligned}\left|\sum_{i=0}^{n-1} f(t_i)(x_{i+1}-x_i) - \frac{1}{2}\right| &= \left|\sum_{i=0}^{n-1}(x_{i+1}-x_i) - \sum_{i=0}^{n-1}\frac{1}{2}(x_{i+1}+x_i)(x_{i+1}-x_i)\right| \\ &= \left|\sum_{i=0}^{n-1}(f(t_i) - \frac{1}{2}(x_{i+1}-x_i))(x_{i+1}-x_i)\right| \\ &\le \sum_{i=0}^{n-1}\frac{1}{2}(x_{i+1}-x_i)^2 \\ &< \frac{\varepsilon}{2}\sum_{i=0}^{n-1}(x_{i+1}-x_i) \\ &= \frac{\varepsilon}{2}.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, f is Riemann integrable on $[0,1]$ and $\int_0^1 f = \frac{1}{2}$.

To see that f is Darboux integrable, note that

$$U_{f,P} = \sum_{i=0}^{n-1}(x_{i+1} - x_i) \sup_{x\in[x_i,x_{i+1}]} f(x) = \sum_{i=0}^{n-1} x_{i+1}(x_{i+1} - x_i)$$

and

$$L_{f,P} = \sum_{i=0}^{n-1}(x_{i+1} - x_i) \inf_{x\in[x_i,x_{i+1}]} f(x) = \sum_{i=0}^{n-1} x_i(x_{i+1} - x_i).$$

Since

$$L_{f,P} < \sum_{i=0}^{n-1} \frac{1}{2}(x_{i+1} + x_i)(x_{i+1} - x_i) < U_{f,P},$$

by repeating the above calculations we get

$$\overline{\int_0^1} f = \frac{1}{2} = \underline{\int_0^1} f.$$

Hence, f is Darboux integrable with $\int_0^1 f = 1$.

3. Note that

$$U_{f,P} = \sum_{i=0}^{n-1}(x_{i+1} - x_i) \sup_{x\in[x_i,x_{i+1}]} f(x) = \sum_{i=0}^{n-1}(x_{i+1} - x_i)\cdot 1 = 1$$

and

$$L_{f,P} = \sum_{i=0}^{n-1}(x_{i+1} - x_i) \inf_{x\in[x_i,x_{i+1}]} f(x) = \sum_{i=0}^{n-1}(x_{i+1} - x_i)\cdot 0 = 1.$$

Hence,

$$\overline{\int_0^1} f = 1 \neq 0 = \underline{\int_0^1} f$$

and f is not integrable.

4. Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $\frac{2}{N} < \varepsilon$. Take $\delta = \varepsilon$, then for any tagged partition $x_0, \cdots, x_n$ and $t_0, \cdots, t_{n-1}$ with mesh $< \delta$, we have

$$\begin{aligned}\left|\sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i) - 0\right| &= \left(\sum_{x_{i+1}\le\frac{1}{N}} + \sum_{x_{i+1}>\frac{1}{N}}\right) f(t_i)(x_{i+1} - x_i) \\ &= \sum_{x_{i+1}\le\frac{1}{N}} 1\cdot(x_{i+1} - x_i) + \sum_{x_{i+1}>\frac{1}{N}} f(t_i)(x_{i+1} - x_i) \\ &\le \frac{1}{N} + \sum_{x_{i+1}>\frac{1}{N}} f(t_i)(x_{i+1} - x_i).\end{aligned}$$

Note that there can only be at most $N-1$ tags with $t_i > \frac{1}{N}$, so

$$\left|\sum_{i=0}^{n-1} f(t_i)(x_{i+1}-x_i) - 0\right| \le \frac{1}{N} + (N-1)\frac{1}{N^2} < \frac{2}{N} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\int_0^1 f = 0$.

5. Let $\varepsilon > 0$ be given. Since f is integrable, we can choose $\delta > 0$ such that for all partitions with mesh $< \delta$, we have

$$U_{f,P} - L_{f,P} < \varepsilon.$$

Hence,

$$\begin{aligned} U_{|f|,P} - L_{|f|,P} &= \sum_{i=0}^{n-1}(x_{i+1}-x_i)\left(\sup_{x\in[x_i,x_{i+1}]} |f|(x) - \inf_{x\in[x_i,x_{i+1}]} |f|(x)\right) \\ &\le \sum_{i=0}^{n-1}(x_{i+1}-x_i)\left(\sup_{x\in[x_i,x_{i+1}]} f(x) - \inf_{x\in[x_i,x_{i+1}]} f(x)\right) \\ &= U_{f,P} - L_{f,P} \\ &< \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $|f|$ is integrable.

By the hint, $H = \frac{1}{2}(f + g + |f-g|)$ and each term is integrable. Since, integrable functions form a vector space, H is integrable.

6.

$$\begin{aligned} S(\alpha f + \beta g, P, \overrightarrow{c}) &= \sum_{i=0}^{n-1}(\alpha f + \beta g)(t_i)(x_{i+1}-x_i) \\ &= \alpha \sum_{i=0}^{n-1} f(t_i)(x_{i+1}-x_i) + \beta \sum_{i=0}^{n-1} g(t_i)(x_{i+1}-x_i) \\ &= \alpha S(f, P, \overrightarrow{c}) + \beta S(g, P, \overrightarrow{c}). \end{aligned}$$

Now, let $\varepsilon > 0$ is given. Let δ_1, δ_2 be chosen corresponding to $\frac{\varepsilon}{|\alpha|+|\beta|}$ according to the definitions that f and g are integrable respectively. Take $\delta = \min\{\delta_1, \delta_2\}$, then for any tagged partition with mesh $< \delta$, we have

$$\begin{aligned} &\left|S(\alpha f + \beta g, P, \overrightarrow{c}) - \left(\alpha \int_a^b f + \beta \int_a^b g\right)\right| \\ &\le |\alpha| \left|S(f, P, \overrightarrow{c}) - \int_a^b f\right| + |\beta| \left|S(g, P, \overrightarrow{c}) - \int_a^b g\right| \\ &< \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\alpha f + \beta g$ is integrable.

7. Let $y_1, \cdots, y_m$ be points such that $f \neq 0$. Let $M = \max_{1\le j\le m} f(y_j)$. Let $\varepsilon > 0$ be given. Set $\delta = \frac{\varepsilon}{mM}$, then for any tagged partition with mesh $< \delta$, we have

$$\left|\sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i)\right| < \sum_{i=1}^{m} M\delta = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\int_a^b f = 0$.

Take $h = f - g$ and apply the previous result to obtain the conclusion.

8. Note that

$$L_{f,P_n} \le T_n(P_n, f) \le U_{f,P_n}.$$

Since f is integrable, given $\varepsilon > 0$ there exists sufficiently large N such that

$$U_{f,P_n} - L_{f,P_n} < \varepsilon.$$

Therefore,

$$T_n(P_n, f) - \underline{\int_a^b} f \le U_{f,P_n} - L_{f,P_n} < \varepsilon$$

and

$$\overline{\int_a^b} f - T_n(P_n, f) \le U_{f,P_n} - L_{f,P_n} < \varepsilon.$$

This shows that $\left|T_n(P_n, f) - \int_a^b f\right| < \varepsilon$ and hence $\lim_{n\to\infty} T_n(P_n, f) = \int_a^b f$.

9. By Q7, F' is integrable and

$$\int_a^b f = \int_a^b F'.$$

We use the same notations as in Q8 except we define

$$T_n(P_n, F) = \sum_{i=1}^{n} F'(\zeta_i)\frac{b-a}{n}$$

where ζ_i is the point such that

$$F(x_i) - F(x_{i-1}) = F'(\zeta_i)(x_i - x_{i-1}).$$

Note that $F(b) - F(a) = T_n(P_n, F)\ \forall n \in \mathbb{N}$. Using the same proof as in Q8 allows as to conclude $\lim_{n\to\infty} T_n(P_n, F) = \int_a^b F'$.

10. Let $\varepsilon > 0$ be arbitrary. Fix $0 \le i \le n-1$, let $y_1, y_2 \in [x_i, x_{i+1}]$ such that

$$\sup_{x \in [x_i, x_{i+1}]} \left(\frac{1}{f}\right)(x) - \varepsilon < \left(\frac{1}{f}\right)(y_1)$$

and

$$inf_{x \in [x_i, x_{i+1}]} \left(\frac{1}{f}\right)(x) + \varepsilon > \left(\frac{1}{f}\right)(y_2).$$

Then,

$$\begin{aligned}
&\sup_{x \in [x_i, x_{i+1}]} \left(\frac{1}{f}\right)(x) - \inf_{x \in [x_i, x_{i+1}]} \left(\frac{1}{f}\right)(x) \\
&< 2\varepsilon + \left(\left(\frac{1}{f}\right)(y_1) - \left(\frac{1}{f}\right)(y_2)\right) \\
&= 2\varepsilon + \frac{f(y_2) - f(y_1)}{f(y_1)f(y_2)} \\
&\le 2\varepsilon + \frac{f(y_2) - f(y_1)}{m^2} \\
&\le 2\varepsilon + \frac{1}{m^2}\left(\sup_{x \in [x_i, x_{i+1}]} f(x) - \inf_{x \in [x_i, x_{i+1}]} f(x)\right).
\end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$\begin{aligned}
U\left(\frac{1}{f}, P\right) - L\left(\frac{1}{f}, P\right) &\le \frac{1}{m^2}\sum_{i=0}^{n-1}(x_{i+1} - x_i)\left(\sup_{x \in [x_i, x_{i+1}]} f(x) - \inf_{x \in [x_i, x_{i+1}]} f(x)\right) \\
&= \frac{1}{m^2}\left(U(f, P) - L(f, P)\right).
\end{aligned}$$

Finally, given $\varepsilon > 0$, since f is integrable, we can choose $\delta > 0$ corresponding to $m^2\varepsilon$ in the definition of f. Then, for all partitions with mesh $< \delta$,

$$U\left(\frac{1}{f}, P\right) - L\left(\frac{1}{f}, P\right) \le \frac{1}{m^2}\left(U(f, P) - L(f, P)\right) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\frac{1}{f}$ is integrable.