1. Consider $\dfrac{f(0+h)-f(0)}{h} = \begin{cases} 2h & \text{if } h\in\mathbb{Q} \\ 0 & \text{if } h\in\mathbb{R}\setminus\mathbb{Q}. \end{cases}$

Hence, if $\varepsilon>0$ is given, then $\forall\ |h|<\delta := \frac{\varepsilon}{2}$, then

$$\left|\frac{f(h)-f(0)}{h} - 0\right| \leq 2|h| < \varepsilon.$$

Thus, $f'(0)=0$.

Note that f is not differentiable for $x\neq 0$ since f is not continuous except at 0.

2. $f(x)=f(-x)$
$f'(x) = -f'(-x)$ using chain rule.

$g(x) = -g(-x)$
$g'(x) = g'(-x)$ using chain rule.

3. By MVT, $|\sin x - \sin y| = |\sin z|\,|x-y|$ for some z lying between x and y
$\leq |x-y|$.

4. Let $\varepsilon>0$ be given. Let $\delta>0$ such that $|f'(a+h)-A|<\varepsilon\ \forall h<\delta$.

Then, $\left|\dfrac{f(a+h)-f(a)}{h} - A\right| = |f'(c)-A|$, where $c\in(a,a+h)$
$< \varepsilon$.

Since $\varepsilon>0$ is arbitrary, $f'(a)=A$.

5. Example : $f : [0,1] \to \mathbb{R}$, $f(x) = \sqrt{x}$.
f is uniformly continuous because f is continuous on a compact interval, but $f'(x) = \frac{1}{2\sqrt{x}}$ for $x \in (0,1]$ and so f' is unbounded on $(0,1)$.

6. Let $x, y \in I$, then $|f(x) - f(y)| = |f'(z)||x-y|$, where z lies between x, y
$\leq M \cdot |x-y|$, where $M = \max_{z \in I} |f'(z)|$.

7. Let $h(x) = |f(x)|^2 + |g(x)|^2 - 1$, then
$$
\begin{aligned}
h' &= 2ff' + 2gg' \\
&= 2fg - 2fg \\
&= 0.
\end{aligned}
$$
Hence, $h \equiv 0$ and $f^2 + g^2 \equiv 1$.

8. Let $\varepsilon > 0$ be given. Let $\delta > 0$ such that $\forall x, y \in I$ with $|x-y| < \delta$, then
$$\left| \frac{f(x) - f(y)}{x-y} - f'(x) \right| < \frac{\varepsilon}{2}.$$

Then, $\forall |x-y| < \delta$, $|f'(x) - f'(y)| = \left| \left(f'(x) - \frac{f(y)-f(x)}{y-x} \right) + \left(\frac{f(x)-f(y)}{x-y} - f'(y) \right) \right|$
$$\leq \left| \frac{f(y)-f(x)}{y-x} - f'(x) \right| + \left| \frac{f(x)-f(y)}{x-y} - f'(y) \right|$$
$$< \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, f' is uniformly continuous.