第 Q1-3 題
(答題不得寫在紅綫外)

第 頁

1. Fix $x_0 \in \mathbb{R}$. For $h \in \mathbb{R}$,

$$|f(x_0+h) - f(x_0)| = |3x_0^2 h + 3x_0 h^2 + h^3|$$
$$= |h|\,|3x_0^2 + 3x_0 h + h^2|$$
$$\leq |h|\,(3x_0^2 + 3|x_0|\,|h| + h^2)$$

Hence, for $\delta = \min\left\{\frac{\varepsilon}{3x_0^2 + 3|x_0| + 1}, 1\right\}$, for $|h| < \delta$,

$$|f(x_0+h) - f(x_0)| < \frac{\varepsilon}{3x_0^2 + 3|x_0| + 1}(3x_0^2 + 3|x_0| + 1)$$
$$= \varepsilon$$

Since $\varepsilon > 0$ and $x_0 \in \mathbb{R}$ is arbitrary, f is continuous.

2. Fix $\varepsilon > 0$. Choose $\delta = \min\{\varepsilon, 1\}$, then for any $|h| < \delta$,

$$|f(0+h) - f(0)| = \begin{cases} |h^2| & \text{if } h > 0 \\ |h| & \text{if } h < 0 \end{cases}$$
$$\leq h < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, f is continuous at 0.

3. Fix $\varepsilon > 0$. Choose $\delta = \varepsilon$, then for any $0 < |h| < \delta$,

$$|f(0+h) - f(0)| = |h \sin \tfrac{1}{h}|$$
$$\leq |h| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, f is continuous at 0.

第 Q4-7 題
(答題不得寫在紅綫外)

第 頁

4. Fix $x_0 \in \mathbb{R}$, $\varepsilon_0 = \frac{1}{2}$.
First suppose $x_0 \in \mathbb{Q}$; let $\delta > 0$ be given. The density of $\mathbb{R} \setminus \mathbb{Q}$ on $\mathbb{R}$ gives $y_0 \in (x_0 - \delta, x_0 + \delta)$ such that $y_0 \in \mathbb{R} \setminus \mathbb{Q}$.
Thus,
$$|f(y_0) - f(x_0)| = 1 \geq \varepsilon_0.$$
Since $\delta > 0$ is arbitrary, f is not continuous at $x_0 \in \mathbb{Q}$.
Suppose $x_0 \in \mathbb{R} \setminus \mathbb{Q}$; let $\delta > 0$ be given. The density of $\mathbb{Q}$ on $\mathbb{R}$ gives $y_0 \in (x_0 - \delta, x_0 + \delta)$ such that $y_0 \in \mathbb{Q}$.
Thus,
$$|f(y_0) - f(x_0)| = 1 \geq \varepsilon_0.$$
This shows that f is not continuous at $x_0 \in \mathbb{R} \setminus \mathbb{Q}$.

5. Since continuous functions is closed under addition and scale multiplication, it suffices to show that $f_n(x) = x^n$ is a continuous function for any $n \in \mathbb{N}$.
For any $x \in \mathbb{R}$, $|h| \leq 1$,
$$|f_n(x+h) - f_n(x)| \leq |h| \sum_{k=0}^{n-1} C_k^n |x|^k.$$
Hence, if we fix $\delta = \min\{1, \frac{\varepsilon}{\sum_{k=0}^{n-1} C_k^n |x|^k}\}$, then
$$|f_n(x+h) - f_n(x)| < \varepsilon \quad \forall\ |h| < \varepsilon.$$
This shows that f_n is continuous and the proof is completed.

6. Let $\varepsilon > 0$ be given and fix $\delta = \varepsilon$, then $\forall\ |h| < \delta$, $x \in \mathbb{R}$
$$|f(x+h) - f(x)| < \varepsilon.$$
This shows that f is continuous.

7a. Let $x_0 \in \mathbb{R} \setminus \mathbb{Q}$. Let $\{x_n\} \subset \mathbb{Q}$ such that $\lim_{n\to\infty} x_n = x_0$. By the continuity of f,
$$f(x_0) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0.$$
b. Set $f = g - h$ and apply Q7(a).

8. Suppose $x_0 \in \mathbb{R}\setminus\mathbb{Q}$. Let $\varepsilon > 0$ be given. Let $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$.
Since $x_0 \notin \mathbb{Q}$, $\delta = \min\{|x_0 - r| : r = \frac{p}{q},\ 0 < p < |q|,\ |q| \leq N,\ p, q \in \mathbb{Z}\} > 0$.

Then $\forall x \in (x_0 - \delta, x_0 + \delta) \cap \mathbb{Q}$, $x = \frac{p}{q}$ for some $|q| > N$, meaning

$$|f(x) - f(x_0)| = \frac{1}{|q|} < \frac{1}{N} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, f is continuous on $\mathbb{R}\setminus\mathbb{Q}$.

Suppose $x_0 \in \mathbb{Q}$: then $f(x_0) > 0$. Set $\varepsilon_0 = \frac{1}{2} f(x_0)$. Let $\delta > 0$ be given. By the density of $\mathbb{R}\setminus\mathbb{Q}$ in $\mathbb{R}$, we can obtain $y_0 \in (x_0 - \delta, x_0 + \delta) \cap \mathbb{R}\setminus\mathbb{Q}$ and

$$|f(y_0) - f(x_0)| = f(x_0) > \varepsilon_0.$$

Since $\delta > 0$ is arbitrary, f is discontinuous on every $x \in \mathbb{Q}$.

9. Set $\varepsilon_c = \frac{1}{2} f(c)$ and choose δ_c corresponding to ε_c, then, $\forall |x - c| < \delta_c$,

$$|f(x) - f(c)| < \varepsilon_c, \quad \text{or} \quad f(x) > f(c) - \varepsilon_c = \frac{1}{2} f(c) > 0.$$

10. Let $x_0 \in \mathbb{R}\setminus\{nT : n \in \mathbb{Z}\}$. Suppose $x_0 = mT + y_0$ for some $m \in \mathbb{Z}$, $y_0 \in (0, T)$. Let $\varepsilon > 0$ and choose δ corresponding to ε using the continuity of f at y_0, then $\forall |x - x_0| < \min\{\delta, y_0, T - y_0\}$

$$|f(x) - f(x_0)| = |f(x - mT) - f(y_0)| < \varepsilon.$$

For $x_0 \in \{nT : n \in \mathbb{Z}\}$, say, $x_0 = mT$ for some $m \in \mathbb{Z}$, let $\varepsilon > 0$ and choose δ_1, δ_2 corresponding to ε using the continuities of f at 0 and T respectively. Then $\forall |x - x_0| < \min\{\delta_1, \delta_2, T\}$

$$|f(x) - f(x_0)| = \begin{cases} |f(x - mT) - f(0)| & \text{if } x > x_0 \\ |f(x - (m-1)T) - f(T)| & \text{if } x < x_0 \end{cases}$$
$$< \varepsilon.$$

10. Since $\varepsilon > 0$ is arbitrary, f is continuous.

11. Let $\varepsilon > 0$ be given and let $y \in \mathbb{R}$. Choose δ corresponding to ε using the continuity of f at x_0, then $\forall\, |x-y| < \delta$,

$$|f(x) - f(y)| = |f(x - (y - x_0)) - f(x_0)| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, f is continuous.

12. We prove by contradiction. Suppose the conclusion is false, then there a sequence $\{x_n\} \subset [a, b]$ such that $f(x_n) < \frac{1}{n}$ $\forall n$.

Now, since $[a, b]$ is compact, there exists a subsequen $\{x_{n_k}\}$ such that $\lim_{n \to \infty} x_n =: x_0$ exists.

Since f is continuous $f(x_0) = \lim_{n \to \infty} f(x_n) = 0$ which is a contradiction.

13. Note that for every odd degree polynomial, we either have

$$\begin{cases} \lim_{x \to \infty} f(x) = \infty \\ \lim_{x \to -\infty} f(x) = -\infty \end{cases} \quad \text{or} \quad \begin{cases} \lim_{x \to \infty} f(x) = -\infty \\ \lim_{x \to -\infty} f(x) = \infty \end{cases}.$$

In particular, there $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) > 0$ and $f(x_2) < 0$. By the intermediate value theorem, there exist x_0 in between x_1 and x_2 such that $f(x_0) = 0$.

14. Consider $g(x) = f(x) - f(x + \frac{1}{2})$, $x \in [0, \frac{1}{2}]$, then g is continuous. Now, $g(\frac{1}{2}) = f(\frac{1}{2}) - f(1) = f(\frac{1}{2}) - f(0) = -(f(0) - f(\frac{1}{2})) = g(0)$.

If $g(0) = g(\frac{1}{2})$, then we are done; otherwise, by intermediate value theorem, $\exists\, c \in (0, \frac{1}{2})$ such that $g(c) = 0$, i.e. $f(c) = f(c + \frac{1}{2})$.

15. h is a continuous function.

Proof: Case 1: $f(x_0) - g(x_0) > 0$

By Q9, we know that $f(x) - g(x) > 0 \ \forall |x - x_0| < \delta_{x_0}$ for some $\delta_{x_0} > 0$. In particular,

$$h(x) = f(x) \quad \forall |x - x_0| < \delta_{x_0}.$$

Hence, $\lim_{x \to x_0} h(x) = \lim_{x \to x_0} f(x) = f(x_0) = h(x_0)$.

Case 2: $f(x_0) - g(x_0) < 0$

It is similar to Case 1.

Case 3: $f(x_0) = g(x_0)$

Let $\varepsilon > 0$. Let δ_1, δ_2 corresponding to ε be picked using the continuities of f and g respectively, then $\forall |x - x_0| < \min\{\delta_1, \delta_2\}$,

$$|h(x) - h(x_0)| = \begin{cases} |f(x) - f(x_0)| & \text{if } f(x) \geq g(x) \\ |g(x) - g(x_0)| & \text{if } g(x) > f(x) \end{cases}$$

$$< \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, f is continuous.

16a Since $f(\mathbb{R})$ is bounded, $\lim_{x \to -\infty} g(x) = -\infty$ and $\lim_{x \to \infty} g(x) = +\infty$, where $g(x) = f(x) - x$. By the intermediate value theorem, g has a zero; or $f(x_0) = x_0$ for some $x_0 \in \mathbb{R}$.

b. Note that $\{a_n\}$ converges since $\{a_n\}$ is increasing and bounded. Taking limits on $a_{n+1} = f(a_n)$, $a = f(a)$ for some $a \in f(\mathbb{R})$.

16c. An approximate solution could be found by the iteration: $\begin{cases} a_1 = 0 \\ a_{n+1} = f(a_n) \end{cases}$

as f and a_1 satisfy all the requirements of Q16(b).

Note: This method does not provide any information for the rate of convergence and error.

17 Let $\{f(x_n)\} \subset f(K)$. Using the compactness of K, there exists $x_0 \in K$ s.t. $\lim_{k\to\infty} x_{n_k} = x_0$. In particular, $\lim_{k\to\infty} f(x_{n_k}) = f(x_0) \in f(K)$.

18. We only show that f attains its absolute max.
Since K is compact and f is continuous, $f(K)$ is bounded.
In particular, $\sup_K f$ exists in $\mathbb{R}$.

Let $\{x_n\} \subset K$ s.t. $f(x_n) > \sup_K f - \frac{1}{n}$.
Since K is compact, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = x_0 \in K$.

Taking limits, $f(x_0) \geq \sup_K f$, or $f(x_0) = \sup_K f$.

19 & 20. We prove the following:
Suppose $f: \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{x\to\infty} f(x), \lim_{x\to-\infty} f(x) \in \mathbb{R}$, then f is uniformly continuous.

Proof: Let $\varepsilon > 0$. Choose M_1, M_2 s.t. $\begin{cases} |f(x) - \lim_{x\to\infty} f(x)| < \frac{\varepsilon}{2} & \forall x > M_1 \\ |f(x) - \lim_{x\to-\infty} f(x)| < \frac{\varepsilon}{2} & \forall x < M_2 \end{cases}$

Hence, $\forall |x|, |y| > M = \max\{|M_1|, |M_2|\}$,
$|f(x) - f(y)| < \varepsilon$.

19&20. Proof: Now, since f is continuous on the compact set $[-M-1, M+1]$, f is uniformly continuous on $[-M-1, M+1]$.
Choose δ using the uniform continuity of f on $[-M-1, M+1]$, then $\forall\ |x-y| < \min\{\delta, 1\}$,
$$|f(x) - f(y)| < \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, f is uniformly continuous on $\mathbb{R}$.

23. Note that there is a typo in "$|f(x)| \geq k \geq 0$" since we requires $k > 0$.
Let $\varepsilon > 0$. Let δ corresponding to $k^2\varepsilon$ be choosen using the uniform continuity of f, then $\forall\ |x-y| < \delta$, $x, y \in A$,
$$\left|\frac{1}{f(x)} - \frac{1}{f(y)}\right| = \left|\frac{1}{f(x)f(y)}\right| |f(x) - f(y)|$$
$$\leq \frac{1}{k^2} |f(x) - f(y)|$$
$$< \varepsilon .$$
Since $\varepsilon > 0$ is arbitrary, f is uniformly continuous.

24. Let $\varepsilon > 0$. Let δ be choosen corresponding to ε using the uniform continuity of f on A. Let $N \in \mathbb{N}$ s.t. $\forall\ m, n \geq N$, $|x_n - x_m| < \delta$.
Then, $\forall\ m, n \geq N$, $|f(x_n) - f(x_m)| < \varepsilon$ as required.

The statement is false if f is only continuous. Consider $f: (0,\infty) \to (0,\infty)$ with $f(x) = \frac{1}{x}$ and $\{x_n\}_{n\in\mathbb{N}} = \{\frac{1}{n}\}_{n\in\mathbb{N}}$, then $\{x_n\}$ is Cauchy in $(0,\infty)$ but $\{f(x_n)\}$ is not.

21. Let T be a period of f, i.e. $f(x+T)=f(x)$.
Consider f as a function on $[0,T]$, then f is uniformly continuous and bounded on $[0,T]$.
Hence, clearly, f is bounded as a function on $\mathbb{R}$.
Now, we show that f is uniformly continuous on $\mathbb{R}$.
Let $\varepsilon>0$ be given. Let δ be chosen corresponding to $\frac{\varepsilon}{2}$ using the uniformly continuity of f on $[0,T]$.
Let $x,y\in\mathbb{R}$ with $|x-y|<T$. Without lose of generality, we may assume $x>y$.
Case 1: $x,y\in[mT,(m+1)T]$ for some $m\in\mathbb{Z}$.
Then if $|x-y|<\delta$, we have,

$$|f(x)-f(y)| = |f(x-mT)-f(y-mT)| < \frac{\varepsilon}{2}.$$

Case 2: $x\in[mT,(m+1)T]$ and $y\in[(m-1)T,mT]$ for some $m\in\mathbb{Z}$.
Then if $|x-y|<\delta$, we have,

$$\begin{aligned}|f(x)-f(y)| &\le |f(x)-f(mT)|+|f(mT)-f(y)| \\ &= |f(x-mT)-f(0)|+|f(T)-f(y-(m-1)T)| \\ &< 2\left(\frac{\varepsilon}{2}\right)=\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, f is uniformly continuous on $\mathbb{R}$.

22. $f:(a,b)\to\mathbb{R}$ is uniformly continuous

$\Rightarrow$ $g:[a,b]\to\mathbb{R}$ is uniformly continuous
where $g(a)=\lim_{x\to a^+}f(x)$, $g(x)=f(x)$ for $a<x<b$ and $g(b)=\lim_{x\to b^-}f(x)$

$\Rightarrow$ g is bounded
$\Rightarrow$ f is bounded

$f((a,b))$ is bounded $\Rightarrow$ g is well-defined and continuous
$\Rightarrow$ g is uniformly continuous
$\Rightarrow$ f is uniformly continuous