1a. Let $\varepsilon > 0$ be given. By the Archimedian Property, $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$.

Then $\forall n \geq N$, $$0 < \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} \leq \frac{1}{N} < \varepsilon,$$

or, $$\left| \frac{n}{n^2+1} - 0 \right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$.

b. Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ s.t. $\frac{2}{N+1} < \varepsilon$.

Then $\forall n \geq N$, $$2 > \frac{2n}{n+1} = 2 - \frac{2}{n+1} > 2 - \varepsilon,$$

or, $$\left| \frac{2n}{n+1} - 2 \right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\lim_{n\to\infty} \frac{2n}{n+1} = 2$.

c. Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ s.t. $\frac{5}{2n^2+3} < \varepsilon$.

Then $\forall n \geq N$, $$\frac{1}{2} > \frac{n^2-1}{2n^2+3} = \frac{1}{2} - \frac{5}{2n^2+3} > \frac{1}{2} - \varepsilon,$$

or $$\left| \frac{n^2-1}{2n^2+3} - \frac{1}{2} \right| < \varepsilon.$$

2. Let $\varepsilon > 0$ be given. Let $[\alpha]$ be the integral part of α.

Choose $N \geq 2[\alpha]+1$ s.t. $\left(\prod_{i=1}^{2[\alpha]} \frac{\alpha}{i}\right)\left(\frac{1}{2}\right)^{N-2[\alpha]-1} < \varepsilon$.

Then $\forall n \geq N$,

$$\frac{\alpha^n}{n!} = \left(\prod_{i=1}^{2[\alpha]} \frac{\alpha}{i}\right)\left(\prod_{j=2[\alpha]+1}^{n} \frac{\alpha}{i}\right)$$

$$\leq \left(\prod_{i=1}^{2[\alpha]} \frac{\alpha}{i}\right) \prod_{j=2[\alpha]+1}^{n} \frac{1}{2}$$

$$= \left(\prod_{i=1}^{2[\alpha]} \frac{\alpha}{i}\right)\left(\frac{1}{2}\right)^{n-(2[\alpha]+1)}$$

$$\leq \left(\prod_{i=1}^{2[\alpha]} \frac{\alpha}{i}\right)\left(\frac{1}{2}\right)^{N-(2[\alpha]+1)}$$

$$< \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\lim_{n\to\infty} \frac{\alpha^n}{n!} = 0$.

3a. Suppose the sequence converges and $\lim_{n\to\infty} a_n = x \in \mathbb{R}$.

Choose $\varepsilon = \frac{1}{2}$, Then $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$|a_n - x| < \frac{1}{2}.$$

When $n \geq N$ is odd, $|x+1| < \frac{1}{2}$.

When $n \geq N$ is even, $|x-1| < \frac{1}{2}$.

But it is impossible for both inequalities to hold. Hence, the sequence must diverge.

3b Suppose the sequence converges and $\lim_{n\to\infty} a_n = x \in \mathbb{R}$.

Pick $\varepsilon = \frac{1}{4}$, then $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$|a_n - x| < \frac{1}{4}.$$

When n is odd, $|n - x| < \frac{1}{4}$.

When n is even, $|x| < \frac{1}{4}$

But it is impossible for both inequalities to hold. Hence, the sequence must be divergent.

4. Define $x_n = y_n = (-1)^n$. Then $\lim_{n\to\infty} (x_n - y_n) = 0$ but both sequences diverge.

5. Define $x_n = y_n = (-1)^n$, then $\lim_{n\to\infty} x_n y_n = 1$ but both sequences diverge.

6. $\lim_{n\to\infty} x_n = 0 \iff \forall \varepsilon > 0, \exists N = N(\varepsilon)$ s.t.

$$|x_n - 0| < \varepsilon \quad \forall n \geq N.$$

$\iff \forall \varepsilon > 0, \exists N = N(\varepsilon)$ s.t.

$$||x_n| - 0| < \varepsilon \quad \forall n \geq N$$

$\iff \lim_{n\to\infty} |x_n| = 0$

For the example pick $x_n = (-1)^n$ then the result follows.

7. Fix $\varepsilon = \frac{1}{2}x$, then $\exists N$ s.t. $\forall n \geq N$,

$$|x_n - x| < \frac{1}{2}x.$$

In particular, $\forall n \geq N$,

$$x_n > x - \frac{1}{2}x = \frac{1}{2}x > 0.$$

8a. Suppose $x < 0$. Then the last question shows that $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $x_n < 0$. This is a contradiction and hence $x \geq 0$.

For the example, put $x_n = \frac{1}{n}$, then $x_n > 0$ $\forall n \in \mathbb{N}$ and $x = 0$.

b. The statement is meaningless if either a_n or b_n diverges. We must assume $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist.

In this case, using (a), $\lim_{n\to\infty}(a_n - b_n) \geq 0$ and hence

$$\lim_{n\to\infty} a_n \geq \lim_{n\to\infty} b_n.$$

9. Let $\varepsilon > 0$ be given and $\lim_{n\to\infty} x_n = x \in \mathbb{R}$. Since $\lim_{n\to\infty} x_n = x$, $\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$,

$$|x_n - x| < \frac{\varepsilon}{2}.$$

By our assumption, $\exists N_2 \in \mathbb{N}$ s.t.

$$|y_n - x_n| < \frac{\varepsilon}{2}.$$

So, if we put $N = \max\{N_1, N_2\}$, then $\forall n \geq N$,

$$|y_n - x| \leq |y_n - x_n| + |x_n - x| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the result follows.

10. Recall that any convergent sequence is bounded. Therefor, we may assume $\exists M>0$ s.t. $\forall n \in \mathbb{N}$, $|a_n| \leq M$.

Let $\varepsilon>0$ be given. Choose $N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$,

$$|a_n - a| < \frac{\varepsilon}{3}.$$

Now, since N_1 is fixed, we can choose $N_2, N_3 \in \mathbb{N}$ s.t. for $n \geq N_2$,

$$\left|\frac{a_1 + \cdots + a_{N_1}}{n}\right| < \frac{\varepsilon}{3},$$

and for $n \geq N_3$, $\left|\frac{N_1}{N} a\right| < \frac{\varepsilon}{3}$.

Hence, for $n \geq \max\{N_1, N_2, N_3\}$,

$$\left|\frac{1}{n}\sum_{i=1}^{n} a_i - a\right| \leq \left|\frac{1}{n}\sum_{i=N_1+1}^{n}(a_i - a)\right| + \left|\frac{1}{n}\sum_{i=1}^{N_1} a_i\right| + \left|\frac{N_1}{n} a\right|$$

$$< \frac{(n-(N_1+1))}{n} \cdot \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$\leq \varepsilon.$$

Since $\varepsilon>0$ is arbitrary, $\lim_{n\to\infty} x_n = a$.

11. Since $|a|<1$, $\exists\ \varepsilon>0$ s.t. $1-a$, $a-(-1)>\varepsilon$.
By the definition of limit, pick $N\in\mathbb{N}$, s.t. $\forall n\geq N$,

$$|a_n-a|<\frac{\varepsilon}{2}$$

In particular, $\forall n\geq N$, $|a_n|<1-\frac{\varepsilon}{2}$,

or, $|a_n^n-0|<(1-\frac{\varepsilon}{2})^n\to 0$ as $n\to\infty$

Now, let $\varepsilon'>0$ be given.
Choose N_2 s.t. $\forall n\geq N_2$, $(1-\frac{\varepsilon}{2})^n<\varepsilon'$.
Then, $\forall n\geq \max\{N_1,N_2\}$, $|a_n^n-0|<(1-\frac{\varepsilon}{2})^n<\varepsilon'$.

Since $\varepsilon'>0$ is arbitrary, $\lim_{n\to\infty} a_n^n=0$.

12a Suppose $\lim_{n\to\infty} x_n=0$.
Let $\varepsilon>0$ be given, choose $N\in\mathbb{N}$ st. $\forall n\geq N$,

$$|x_n-0|<\sqrt{\varepsilon},$$

or $|x_n^2-0|<\varepsilon$.

Since $\varepsilon>0$ is arbitrary, $\lim_{n\to\infty} x_n^2=0$.

Suppose $\lim_{n\to\infty} x_n^2=0$.
Let $\varepsilon>0$ be given, choose $N\in\mathbb{N}$ st $\forall n\geq N$,

$$|x_n^2-0|<\varepsilon^2,$$

or $|x_n-0|<\varepsilon$.

Since $\varepsilon>0$ is arbitrary, $\lim_{n\to\infty} x_n=0$.

12b. Note that $0 \le a_n^2, b_n^2 \le a_n^2 + b_n^2$.
Since $\lim_{n\to\infty} 0 = \lim_{n\to\infty} (a_n^2 + b_n^2) = 0$, $\lim_{n\to\infty} a_n^2$, $\lim_{n\to\infty} b_n^2$ exist and both equal to 0.
By (a), we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$.

13 Consider $a_{n+2} - a_{n+1} = \frac{a_{n+1}+1}{a_{n+1}+2} - \frac{a_n+1}{a_n+2}$

$$= \frac{a_{n+1} - a_n}{(a_{n+1}+2)(a_n+2)}$$

$$= \cdots$$

$$= \left(\prod_{i=1}^{n} \frac{1}{(a_{i+1}+2)(a_i+2)}\right)(a_2 - a_1)$$

$$= -\frac{1}{3}\prod_{i=1}^{n} \frac{1}{(a_{i+1}+2)(a_i+2)}.$$

Hence, $\{a_n\}$ is a decreasing sequence. Since we have clearly $a_n > 0$ $\forall n$, $\{a_n\}$ is convergent.

Now, taking limits on both sides,

$$a_{n+1} = \frac{a_n+1}{a_n+2}$$

$$a^2 + 2a = a + 1$$

$$a = \frac{-1 + \sqrt{1-4(1)(-1)}}{2} = \frac{-1+\sqrt{5}}{2} \quad (\text{Since } a_n > 0 \ \forall n \Rightarrow a \geq 0).$$

14. Consider $x_{n+2} - x_{n+1} = \frac{1}{2}(x_{n+1} - x_n)$

$$= \cdots$$
$$= \left(\frac{1}{2}\right)^n (x_2 - x_1)$$
$$= -\left(\frac{1}{2}\right)^{n-1}.$$

Hence $\{x_n\}$ is decreasing. Since we clearly have $x_n > 0 \ \forall n$, $\{x_n\}$ is convergent.
Taking limits on both sides,

$$x_{n+1} = \frac{1}{2}x_n + 2$$
$$x = \frac{1}{2}x + 2$$
$$x = 4.$$

15. We show that it is bounded and its convergence will follow from monotone convergence theorem.
For any $n \in \mathbb{N}$, $\sum_{i=1}^{n} \frac{1}{i!} \le 1 + \sum_{i=1}^{n-1} \frac{1}{2^i} = 1 + \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} < 3$.

16. Let $M > 0$ s.t. $|b_n| \le M \ \forall n \in \mathbb{N}$. We show that it is bounded.

$$|S_n| \le \sum_{i=1}^{n} |b_n| r^n$$
$$\le M \sum_{i=1}^{n} r^n$$
$$= Mr \frac{1-r^n}{1-r}$$
$$< \frac{Mr}{1-r}.$$

17. For any $n \in \mathbb{N}$, $\sin(n+2) - \sin n = 2\cos(n+1)\sin 1$.
In particular, to show that $\{\sin n\}_{n\in\mathbb{N}}$ is divergent, it suffices to show that $\forall N \in \mathbb{N}$, $\exists n \ge N$ s.t. $|2\cos(n+1)| > 2\cos\frac{\pi}{3}$.

Note that it is always possible because $|\cos n| \le \cos\frac{\pi}{3} \iff |n - \cancel{2}(k\pi + \frac{\pi}{2})| < \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$,
so $|2\cos(n+1)| > 2\cos\frac{\pi}{3}$ whenever $n+1 \in (\cancel{0} - \frac{\pi}{3}, \frac{\pi}{3}) \cup (\frac{2\pi}{3}, \frac{4\pi}{3}) \cup \cdots$.

17. However, for each interval in the union, its length is $\frac{2\pi}{3} > 1$, so it is impossible that for a given N, we have $n \notin (-\frac{\pi}{3}, \frac{\pi}{3}) \cup (\frac{2\pi}{3}, \frac{4\pi}{3}) \cup \cdots$ for every $n \geq N$. Hence, for each N, we can always find $n \geq N$ such that the above difference $> \cos\frac{\pi}{3}$ ~~sin~~ 1 and therefore $\{\sin n\}_{n\in\mathbb{N}}$ is not Cauchy.

18. Consider the sequence $\{\sin n\}_{n\in\mathbb{N}}$, then this sequence has no limit.

19. Suppose A is compact.
We first show that A is closed, that is, if $\{x_n\}_{n\in\mathbb{N}} \subseteq A$ and it has a limit $x\in\mathbb{R}$, then $x \in A$.
Now, since A is compact, $\exists\, y \in A$ s.t. $\lim\limits_{n\to\infty} x_n = y$.
By the uniqueness of limits, $x = y \in A$.

Next, we prove that A is bounded.
Suppose on the contrary that $\exists \{x_n\} \subseteq A$ s.t. $\forall M \geq 0$, $\exists N$ s.t. $|x_N| > M$.
We construct a subsequence without limit as follows.
Set $y_1 = x_1$, and for each $y_n = x_{m_n}$ defined, we choose an m_n such that

$$\begin{cases} |x_{m_{n+1}}| > |x_{m_n}| + 1 \\ m_{n+1} > m_n \end{cases}.$$

Then $\{m_n\}_{n\in\mathbb{N}}$ is strictly increasing and so $\{y_n\}_{n\in\mathbb{N}}$ is a subsequence of $\{x_n\}_{n\in\mathbb{N}}$. Also, we have

$$|x_{m_{n+1}} - x_{m_n}| > 1 \quad \forall n \in \mathbb{N}.$$

Therefore $\{y_n\}_{n\in\mathbb{N}}$ is not Cauchy.

Suppose A is closed and bounded.
Let $\{x_n\}_{n\in\mathbb{N}} \subseteq A$ be a sequence in A. By the Bolzano-Weierstrass Theorem, $\exists$ a subsequence $\{y_n\}_{n\in\mathbb{N}} \subseteq \{x_n\}$ and a point $x \in A$ s.t. $\lim\limits_{n\to\infty} y_n = x$.

20. We show that $\{x_n\}$ converges to 0.
Let $\varepsilon > 0$, then $\exists N$ s.t. $\forall m, n \geq N$, $|(-1)^m x_m - (-1)^n x_n| < \varepsilon$.
In particular, $\forall n \geq N$, $|x_{n+1} + x_n| < \varepsilon$.
Since $x_n \geq 0$ $\forall n \in \mathbb{N}$, $0 \leq x_n < \varepsilon$ $\forall n \geq N$.
Since this is true $\forall \varepsilon > 0$, we conclude that $\lim_{n \to \infty} x_n = 0$.

21a. Let $\varepsilon > 0$ be given. We choose N s.t. $\frac{1}{N} < \varepsilon$, then $\forall m, n \geq N$,

$$\left| \frac{m+1}{m} - \frac{n+1}{n} \right| = \left| \frac{1}{m} - \frac{1}{n} \right| < \frac{1}{N} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\{\frac{n+1}{n}\}_{n \in \mathbb{N}}$ is Cauchy.

b. Let $\varepsilon > 0$ be given. We choose N s.t. $\frac{1}{2^{N-1}} < \varepsilon$, then $\forall m > n \geq N$,

$$\begin{aligned}
\left| \sum_{i=1}^{m} \frac{1}{i!} - \sum_{i=1}^{n} \frac{1}{i!} \right| &= \sum_{i=n+1}^{m} \frac{1}{i!} \\
&\leq \sum_{i=n+1}^{m} \frac{1}{2^{i-1}} \\
&= \frac{1}{2^n} \sum_{i=0}^{m-n-1} \frac{1}{2^i} \\
&= \frac{1}{2^n} \frac{1 - (\frac{1}{2})^{m-n}}{1 - \frac{1}{2}} \\
&= \frac{1}{2^{n-1}} \left(1 - \left(\frac{1}{2}\right)^{m-n} \right) \\
&< \frac{1}{2^{n-1}} < \varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary $\{\sum_{i=1}^{n} \frac{1}{i!}\}_{n \in \mathbb{N}}$ is Cauchy.

22. Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded, $x := \sup_{n\in\mathbb{N}} x_n \in \mathbb{R}$.

Now, let $\varepsilon > 0$, then $\exists N \in \mathbb{N}$ s.t. $x - \varepsilon < x_N$.

Since $\{x_n\}_{n\in\mathbb{N}}$ is increasing, $\forall m > n \geq N$,

$$\begin{aligned} |x_m - x_n| &= x_m - x_n \\ &< x_m - x_N \\ &\leq x - x_N \\ &< \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy.

23. Let $\varepsilon > 0$; choose N s.t. $\frac{r^n}{1-r} < \varepsilon$.
Then $\forall m > n \geq N$,

$$\begin{aligned} |x_m - x_n| &\leq |x_{n+1} - x_n| + \cdots + |x_m - x_{m-1}| \\ &\leq r^n + \cdots + r^{m-1} \\ &= r^n \frac{1 - r^{m-n}}{1-r} \\ &< \frac{r^n}{1-r} < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy.

24. For $m > n$,

$$\begin{aligned} |x_m - x_n| &\leq |x_{n+1} - x_n| + \cdots + |x_m - x_{m-1}| \\ &\leq C|x_n - x_{n-1}| + \cdots + C|x_{m-1} - x_{m-2}| \\ &\leq \cdots \\ &\leq C^{n-1}|x_2 - x_1| + \cdots + C^{m-1-1}|x_2 - x_1| \\ &= C^{n-1}|x_2 - x_1|(1 + \cdots + C^{m-n-1}) \\ &= \frac{C^{n-1}}{1-C}(1 - C^{m-n})|x_2 - x_1| \\ &< \frac{C^{n-1}}{1-C}|x_2 - x_1|. \end{aligned}$$

Hence, given $\varepsilon > 0$, if we choose $N \in \mathbb{N}$ s.t. $\frac{C^{N-1}}{1-C}|x_2 - x_1| < \varepsilon$,
then $|x_m - x_n| < \varepsilon$ $\forall m > n \geq N$. This shows that $\{x_n\}$ is Cauchy

25. Fix $x \in \mathbb{R}$. Fix N such that $2x < N$, then $\forall\, n > m \geq N$

$$|f_n(x) - f_m(x)| = \left| \sum_{i=m+1}^{n} \frac{x^i}{i!} \right| < \left| \left(\frac{1}{2}\right)^{m+1} \sum_{i=m+1}^{n} \left(\frac{1}{2}\right)^{i-(m+1)} \right|$$

$$\leq \left(\frac{1}{2}\right)^m$$

Let $\varepsilon > 0$ be given.

Chose N' s.t. $\frac{1}{2^{N'}} < \varepsilon$. Then if we choose $N'' = \max\{N, N'\}$, $\forall\, n > m \geq N''$,

$$|f_n(x) - f_m(x)| < \left(\frac{1}{2}\right)^m < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, for a fixed $x \in \mathbb{R}$, $\{f_n(x)\}_{n \in \mathbb{N}}$ is Cauchy.