Q1. $(\mathbb{Z}^+, \mathcal{T})$ $\mathcal{T} = \{ B_n \mid n = 1, 2, 3, \cdots \} \cup \{\mathbb{Z}^+\}$

where $B_n = \{ k \in \mathbb{Z}^+ \mid k < n \}$ for $n = 1, 2, \cdots$

(a). Does $(\mathbb{Z}^+, \mathcal{T})$ satisfy the Bolzano-Weierstrass property?

(b). Find dense $D \subsetneq \mathbb{Z}^+$.

(c). Find an infinit subset $M \subset \mathbb{Z}^+$ which Nowhere dense.

(a). Yes.

For $\forall$ infinite subset $A \subset \mathbb{Z}^+$

We can find $m, n \in A$, st $m > n$

Then m is a cluster pt of A.

since $\forall$ open set contains m

must be of the form:

$B_N = \{1, 2, \cdots, N\}$, where $N > m$

or $\mathbb{Z}^+$

But $n \in (B_N - \{m\}) \cap A \Rightarrow \neq \emptyset$

$n \in (\mathbb{Z}^+ - \{m\}) \cap A \Rightarrow \neq \emptyset$

b). $\forall$ D contains 1 is dense.

Since $\forall$ nonempty open set is of the form $B_N = \{1, 2, \cdots, N\}$, $N \geq 1$

$\Rightarrow B_N \cap D \neq \emptyset$

or $\mathbb{Z}^+$

$\Rightarrow \mathbb{Z}^+ \cap D \neq \emptyset$.

c). $\forall$ M not contains 1 is nowhere dense

Pf: $\overline{M} \subseteq \mathbb{Z}^+ \setminus \{1\} \leftarrow$ closed.

$(\overline{M})^\circ \subseteq (\mathbb{Z}^+ \setminus \{1\})^\circ = \emptyset$

$\forall$ non-empty open set contains $\{1\}$

Q2. If $f_1 \simeq g_1 : X \to Y_1$

$f_2 \simeq g_2 : X \to Y_2$

Show: $(f_1, f_2) \simeq (g_1, g_2) : X \to Y_1 \times Y_2$

Pf: $f_1 \overset{H_1(x,t)}{\simeq} g_1$

$f_2 \overset{H_2(x,t)}{\simeq} g_2$

$\Longrightarrow (f_1, f_2) \simeq (g_1, g_2)$

$(H_1(x,t), H_2(x,t))$

Q3. Let Y be a topo space.

$CY \triangleq Y \times [0,1] / \sim$

where $(y_1, t_1) \sim (y_2, t_2)$

iff $y_1 = y_2, t_1 = t_2$ or $t_1 = t_2 = 1$

Show: CY is contractible

Pf: Choose $\forall y_0 \in Y$

Let $p = [(y_0, 1)]$ & $\underset{\overset{\parallel\!\!|}{p}}{C_p} : CY \to CY$

We have $id_{CY} \overset{\tilde{H}}{\simeq} C_p$ where:

$$\tilde{H}([(y,s)],\ t) = [(y,\ (1-t)s + t)]$$

because:

Claim: $\tilde{H}$ is well defined & cts

Pf of Claim:

- Easy to get $\tilde{H}$ is well defined

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$$\begin{tikzcd}
((y,s),t) \arrow[d, mapsto] & Y \times I \times I \arrow[r, "q \times id_I"] \arrow[d, "H"] & (Y \times I/\sim) \times I \arrow[d, "\tilde{H}"] \\
(y, (1-t)s+t) & Y \times I \arrow[r, "q"] & Y \times I/\sim
\end{tikzcd}$$

If U is open in $Y \times I/\sim$

then

$$(q \times id)^{-1}(\tilde{H}^{-1}(U)) = \underbrace{H^{-1}(q^{-1}(U))}_{\text{open}}$$

$\Rightarrow$ $\tilde{H}^{-1}(U)$ is open

Q4. $S^{n-1} \simeq \mathbb{R}^n \setminus \{0\}$

Let $i: S^{n-1} \to \mathbb{R}^n \setminus \{0\}$

be the inclusion

$$r: \mathbb{R}^n / \{0\} \longrightarrow S^{n-1}$$

$$\vec{x} \longmapsto \frac{\vec{x}}{\|\vec{x}\|}$$

① $r \circ i = id_{S^{n-1}} \simeq id_{S^{n-1}}$

② $i \circ r \overset{H}{\simeq} id_{\mathbb{R}^n \setminus \{0\}}$

$$H: \mathbb{R}^n \setminus \{0\} \times [0,1] \longrightarrow \mathbb{R}^n \setminus \{0\}$$

$$(\vec{x}, t) \longmapsto t\vec{x} + (1-t)\frac{\vec{x}}{|\vec{x}|}$$

. H is well defined.

If NOT, then $t\vec{x} + (1-t)\frac{\vec{x}}{\|\vec{x}\|} = 0$

for some $\vec{x} \in \mathbb{R}^n \setminus \{0\}$, $t \in [0,1]$

$$\Rightarrow \quad t\vec{x} = (t-1)\frac{\vec{x}}{\|\vec{x}\|} \quad ③$$

take norm on both sides

$$\Rightarrow \quad t\|\vec{x}\| = 1-t \quad ④$$

Plug ④ into ③ $\Longrightarrow$ $t\vec{x} = -t\vec{x}$

$\overset{\vec{x} \neq 0}{\Longrightarrow}$ $t = 0$

$\Longrightarrow$ $\vec{0} = 0 \cdot \vec{x} + (1-0)\dfrac{\vec{x}}{\|\vec{x}\|} = \dfrac{\vec{x}}{\|\vec{x}\|}$

$\Longrightarrow$ $\vec{x} = 0$ Contradiction !

- H is cts with $H(\vec{x}, 0) = i \circ r$

$H(\vec{x}, 1) = id$