Lect14-20180305

Mar 01, Thursday, 2018 3:31 PM

Definition. Given X, Y and $A \subset X$ and $f: A \longrightarrow Y$ be continuous. Define

$$X \cup_f Y = (X \sqcup Y)/\sim$$ where

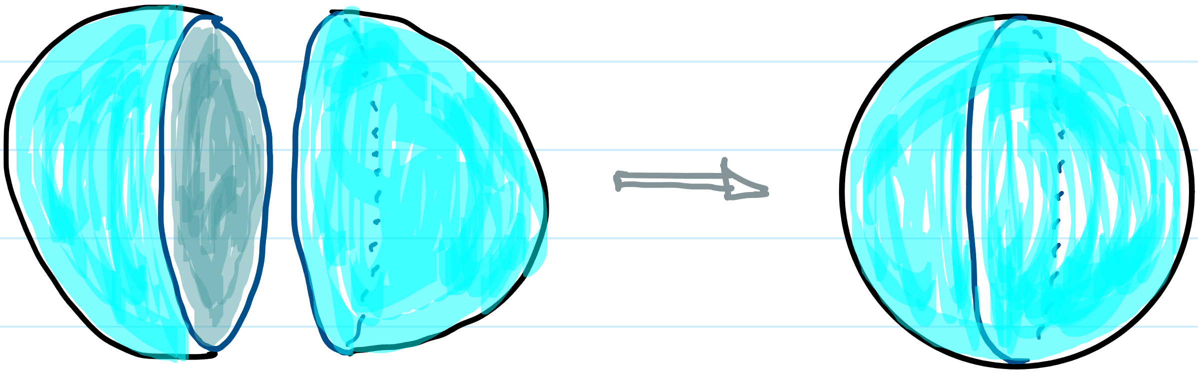
$a \in A$ and $f(a) \in Y$ are identified.

This is called attaching X to Y along $f: A \longrightarrow Y$

Example.

- $X = Y = \mathbb{D}^2$, $A = S^1 \subset X$, $f: S^1 \longrightarrow \mathbb{D}^2$ is the inclusion map, $f(z) = z$

What is $X \cup_f Y$?



- $X = \mathbb{D}^2$, $A = \mathbb{S}^1 \subset X$, $Y = \{y_0\}$

  $f : A \longrightarrow Y$ is the constant map.

  What is $X \cup_f Y$?

  The same as $\mathbb{D}^2 \Big/ \begin{pmatrix} z_1 \sim z_2 \text{ if} \\ z_1, z_2 \in \mathbb{S}^1 \end{pmatrix}$

  $\parallel$

  叉燒包 without meat

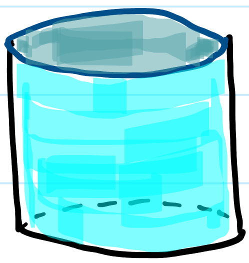
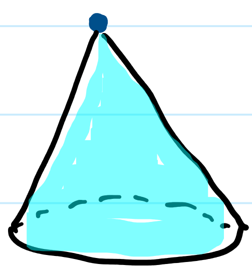
- Let $X = [-1, 1]$, $A = \{x \neq 0\} \subset X$, and

  $f : A \longrightarrow X$, $f(x) = x$.

  Then $X \cup_f X =$ 

- Let $X = \mathbb{S}^1 \times [0, 1]$, $A = \mathbb{S}^1 \times \{1\} \subset X$, $Y = \{0\}$

  and $f : A \longrightarrow Y$ (obviously, constant map)

 $\Longrightarrow$ $X \cup_f Y =$ 

Lect14-p3

Wednesday, 28 February 2018 2:05 PM

Example.

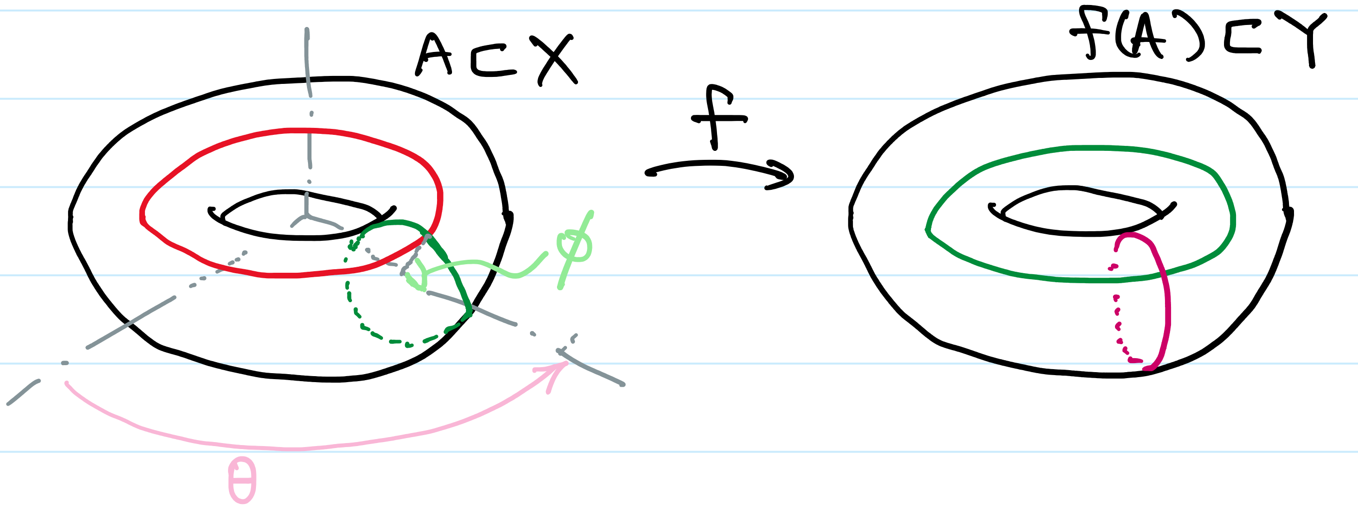
$$S^3 = \{ x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \|x\| = 1 \}$$

$$\| \quad \mathbb{R}^3 \cup \{\infty\} \quad \text{just like} \quad S^2 = \mathbb{C} \cup \{\infty\}$$

by stereographic projection

Solid Torus, $S^1 \times D^2 = X \supset A = S^1 \times S^1$, torus

$\cong Y$



$$(e^{i\theta}, e^{i\phi}) \in S^1 \times S^1 \longmapsto (e^{i\phi}, e^{i\theta}) \in Y$$

$$\cap$$

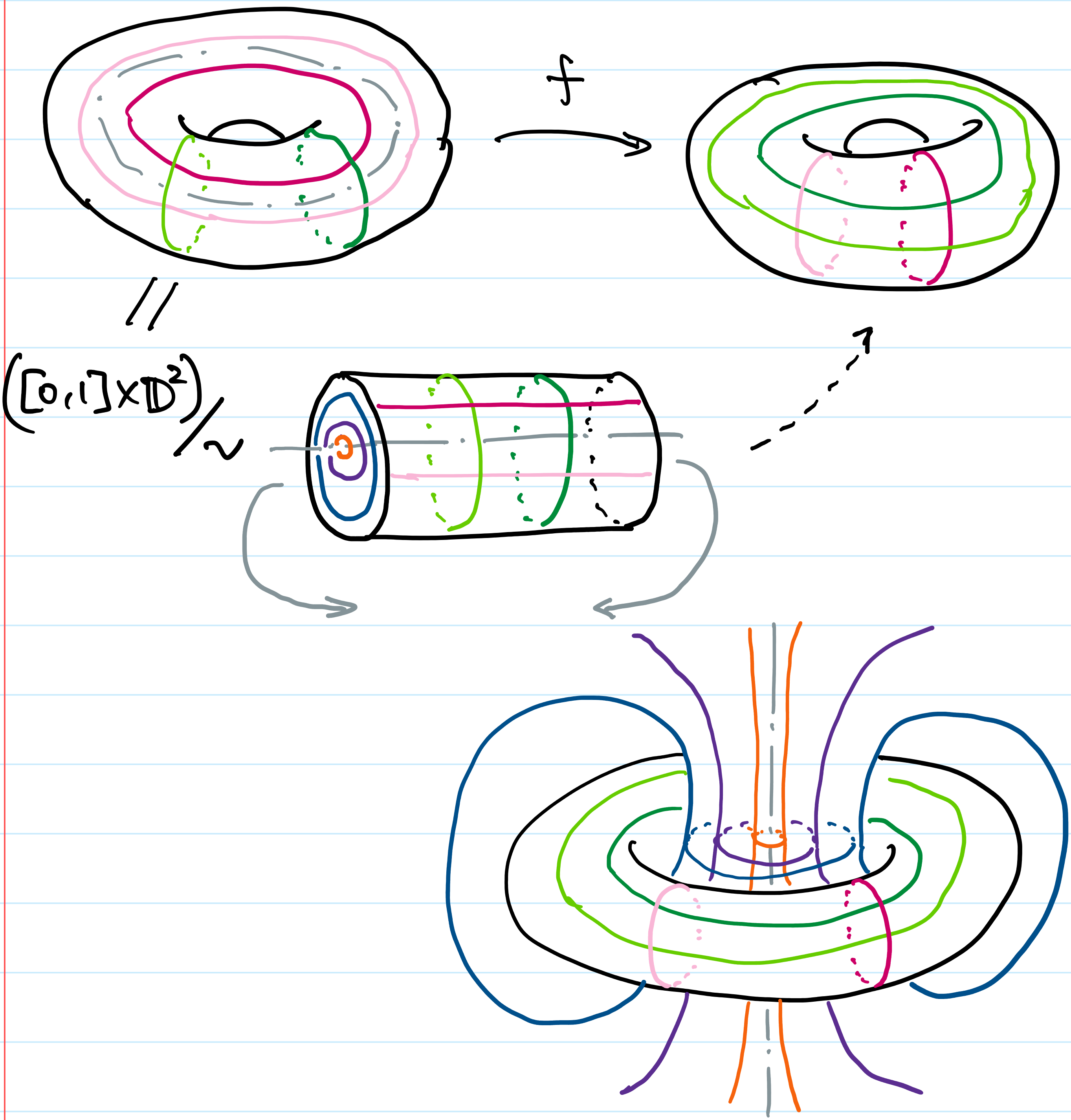
$$S^1 \times D^2 = \{ (e^{i\theta}, re^{i\phi}) : 0 \le r \le 1, \cdots \}$$

What is the space $(S^1 \times D^2) \cup_f (S^1 \times D^2)$?

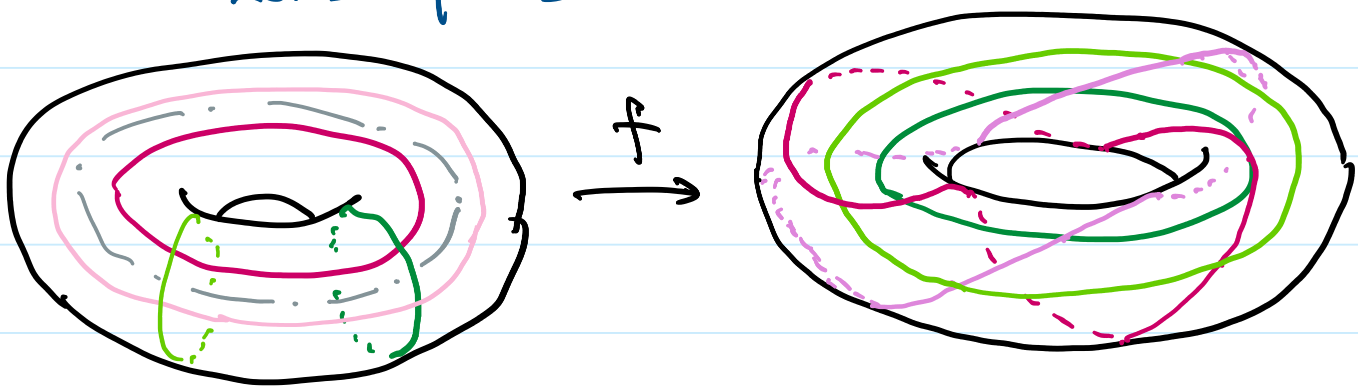
$$(S^1 \times D^2) \cup_f (S^1 \times D^2) = S^3 = \mathbb{R}^3 \cup \{\infty\}$$

Lect14-p4

Mar 05, Monday, 2018 6:56 PM



Other Lens Spaces



Properties of Quotient Topology

Setting: A topological space $(X, \mathcal{T}_X)$

An equivalence relation $\sim$ on X, or

a surjective mapping $q: X \longrightarrow Q$

Result: $(X/\sim, \mathcal{T}_q)$ or $(Q, \mathcal{T}_q)$

What can we say about

$q: (X, \mathcal{T}_X) \longrightarrow (X/\sim \text{ or } Q, \mathcal{T}_q)$?

QT1: q is continuous.

$q^{-1}(V) \in \mathcal{T}_X \xleftarrow{\ q\ }$ For any $V \in \mathcal{T}_q$

$\longleftarrow$ Really, definition of $\mathcal{T}_q$

Is there another $\mathcal{J}$ on $X/\sim$ or Q with

$q: (X, \mathcal{T}_X) \longrightarrow (X/\sim \text{ or } Q, \mathcal{J})$ continuous?

Obviously YES. How to compare $\mathcal{T}_q$ and $\mathcal{J}$?

QT2. $\mathcal{T}_q$ is maximal, i.e., $\mathcal{J} \subset \mathcal{T}_q$.

Lect14-p6

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What do you guess about QT3?

Which one is it about?

$$\text{Any } W \xrightarrow{f} X/\sim \quad \uparrow q \quad X \qquad \text{or} \qquad \text{Any } X/\sim \xrightarrow{g} Z \text{ ?} \quad \uparrow q \quad X$$

QT3. Any $g : (X/\sim \text{ or } Q, \mathcal{T}_q) \longrightarrow (Z, \mathcal{T}_Z)$ is continuous $\Leftrightarrow$ $g \circ q : (X, \mathcal{T}_X) \to (Z, \mathcal{T}_Z)$ is continuous.

" $\Rightarrow$ " Trivial

Composition of continuous mappings

$$(X, \mathcal{T}_X) \xrightarrow{q} (X/\sim \text{ or } Q, \mathcal{T}_q) \xrightarrow{g} (Z, \mathcal{T}_Z)$$

" $\Leftarrow$ " Let $W \in \mathcal{T}_Z$

Need: $\underbrace{g^{-1}(W) \in \mathcal{T}_q}$

How to verify this?

see if $q^{-1}(g^{-1}(W)) \in \mathcal{T}_X$

$$\| $$

$(g \circ q)^{-1}(W)$ Yes, continuity of $g \circ q$

Obvious, next question, is there $\mathcal{T}$ on $X/\sim$ such that any $g : (X/\sim, \mathcal{T}) \longrightarrow (Z, \mathcal{T}_Z)$ is continuous $\Longleftrightarrow$ $g \circ q$ is so ?

Answer. Yes, and

QT4 $\mathcal{T}_q$ is the minimal.

Take the test case of

$$(Z, \mathcal{T}_Z) = \boxed{(X/\sim, \mathcal{T}_q)} \text{ and}$$

$$g = id : (X/\sim, \mathcal{T}) \longrightarrow \boxed{(X/\sim, \mathcal{T}_q)}$$

In this case, g is continuous $\Longleftrightarrow$ $\mathcal{T}_q \subset \mathcal{T}$.

The analogue

Product: PT1, PT2, PT3, PT4 ($\mathcal{T}_\pi$ and π_α)

Quotient: QT1, QT2, QT3, QT4 ($\mathcal{T}_q$ and q)

Definition. A topological space $(X, \mathcal{T})$ is compact if every open cover has a finite subcover.

Precise wordings.

Open cover for X: $\mathcal{G} \subset \mathcal{T}$ with $\underbrace{\bigcup \mathcal{G}}_{\text{The union of all sets in } \mathcal{G}} = X$

Finite subcover of $\mathcal{G}$: A finite $\mathcal{F} \subset \mathcal{G}$ which is a cover for X, i.e., $\bigcup \mathcal{F} = X$.

Example. $\mathbb{R}$ is not compact

$\mathcal{G} = \{(k, k+2) : k \in \mathbb{Z}\}$, $\bigcup \mathcal{G} = \mathbb{R}$

Similarly, every $\mathbb{R}^n$, $n \geq 1$, is non-compact

Example. $(0, 1]$ is not compact

$\mathcal{G} = \{(\frac{1}{k}, 1] : 1 < k \in \mathbb{Z}\}$, $\bigcup \mathcal{G} = (0, 1]$.

Example. Every interval $[a, b]$ is compact.

How do you know?

恭孚 said so! □

Lect14-p9

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Idea 1: Define $L \subset [a,b]$ to be

$\{x \in [a,b] : \mathcal{C} \text{ has a finite subcover for } [a,x]\}$

- L has an upper bound $b \in \mathbb{R}$
- Thus, $s = \sup L \le b$ exists

What happens if $s < b$?

⋮

contradiction !

Cons: This method needs order, so even not valid for $[a,b]^n$, $n > 1$.

Lect14-p10

Sunday, 4 March 2018 3:37 PM

Idea 2: Assume $\mathcal{G}$ has no finite subcover

Subdivide $[a,b] = [a, \frac{a+b}{2}] \cup [\frac{a+b}{2}, b]$

At least one side cannot be covered by any finite subset of $\mathcal{G}$.

Continue to subdivide, by assumption, the process will not stop.

Get nested closed intervals, $F_n \supset F_{n+1}$ with $\operatorname{diam}(F_n) \to 0$.

Thus, $\bigcap_{n=1}^{\infty} F_n = \{x_0\} \subset [a,b]$

⋮

contradiction

Pros. No order is needed, valid for $[a,b]^n$

Cons Seems to be a consequence of complete metric space !

Lect14-p11

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Example. $\mathbb{R}^n$ is a complete metric space but non-compact.

Crucial property in the proof, some sort of "finite size". A totally bounded complete metric space is always compact

Example. There must be a compact space without metric!

Can you find a compact metric space that is not complete?

Answer can be seen below.

Notions of compactness

Heine-Borel

Every open cover for X has a finite subcover

Bolzano-Weierstrass

Every infinite set has a cluster point in X.

Sequentially compact

Every sequence has a convergent subsequence.

Theorem. Equivalent in a separable metric space (thus, it is second countable and Hausdorff).