Lect07-20180129

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Uniqueness Theorem. Given X and Hausdorff Y, $A \subset X$ where A is dense, and continuous mappings $f, g : X \longrightarrow Y$ such that $f|_A \equiv g|_A$. Then $f \equiv g$ on X.

Proof. Start: Let $x \in X$ Wish: $f(x) = g(x)$

What if not, i.e., $f(x) \neq g(x)$

Y is Hausdorff, therefore

$\exists\ V_1, V_2 \in \mathcal{T}_Y$ such that

$f(x) \in V_1$, $g(x) \in V_2$, $V_1 \cap V_2 = \emptyset$

Both $f, g : X \longrightarrow Y$ are continuous

$\therefore \quad U = f^{-1}(V_1) \cap g^{-1}(V_2) \in \mathcal{T}_X$

Also, $x \in U$, i.e., $U \neq \emptyset$; then what

Since A is dense, $\exists\ a \in A \cap U \neq \emptyset$

$f(a) = g(a) \in f(U) \cap g(U) \subset V_1 \cap V_2 = \emptyset$

contradiction

Recall $f: X \to Y$ continuous $\iff$

$$\forall E \subset X,\ f(\overline{E}) \subset \overline{f(E)}$$

any $\downarrow$

$$x_0 \xrightarrow{f} f(x_0) \in V \in \mathcal{T}_Y$$

$x_0 \leftarrow$ pre-image

near

$\cap$ $\qquad$ $f(e)$

$U = f^{-1}(V) \in \mathcal{T}_X$ $\qquad$ $\cap$

$\Downarrow x_0 \in \overline{E}$ $\qquad$ $V \cap f(E)$

near

$\exists\ e \xrightarrow{f} f(e)$

$\cap$

$U \cap E \neq \phi$

In analysis, there is a similar statement

What is it ?

In analysis, we have

$$f(\lim x_n) = \lim f(x_n)$$

or $\quad x_n \to x \implies f(x_n) \to f(x)$

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Definition. A sequence in X is

$$n \in \mathbb{N} \longmapsto x_n \in X$$

Denoted by $(x_n)_{n=1}^{\infty}$

It converges to $x \in X$; or x is a limit; denoted $x_n \to x$ or $\lim_{n\to\infty} x_n = x$ if

$\forall$ $U \in \mathcal{T}$ with $x \in U$, $\exists$ $N \in \mathbb{N}$ such that

$$\forall\ n \geq N \quad x_n \in U$$

(i) a nbhd of x

(ii) $U \in \mathcal{U}_x$, local base of x

First Theorem about limit (What is it?)

Limit is unique if X is Hausdorff

Idea. Assume $x_n \to x$, $x_n \to y$, and $x \neq y$

Wish. Get a contradiction

By $x \neq y$

$\exists$ $U_1, U_2 \in \mathcal{T}$, $x \in U_1$, $y \in U_2$, $U_1 \cap U_2 = \emptyset$

For sufficienty large n,

$x_n \in U_1$ and $x_n \in U_2$

contradiction.

Proposition. For space X and $A \subset X$, if $(x_n)_{n=1}^{\infty}$ in A and $x_n \to x$ then $x \in \overline{A}$

Idea. Consider $U \in \mathcal{T}$ with $x \in U$

Wish: $U \cap A \neq \emptyset$

Simply because for large n,

$$\underset{\substack{\rotatebox{90}{$\in$} \\ A}}{x_n} \in U$$

Corollary. Any convergent sequence in a closed set A always has its limit in A.

Question. Given $x \in \overline{A}$, can we conclude $\exists\ (x_n)_{n=1}^{\infty}$ in A such that $x_n \to x$?

True if $X = \mathbb{R}^n$ with $\mathcal{T}_{std}$.

How to prove it?

By choosing $x_n \in B(x, \frac{1}{n}) \cap A$

True if X is a metric space.

Exercise. True if X is 1st countable.

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Now, 路人甲乙丙 司馬昭

Question. Find a counter-example when X is not 1st countable

Bad Example. $(\mathbb{R}, \text{co-countable})$, $A = \mathbb{R} \setminus \{0\}$

(i) $0 \in \overline{A}$ Why?

Only closed sets are ϕ, countable, and $\mathbb{R}$

(ii) $x_n \in \mathbb{R}\setminus\{0\}$ with $x_n \longrightarrow 0$ gives contradiction.

How?

Let $U = \mathbb{R} \setminus \{1\}$. Then $0 \in U \in \mathcal{T}$

$\exists\, N \;\forall n \geq N \quad x_n \in \mathbb{R}\setminus\{1\}$

Take $V = U \setminus \{x_n : n \geq N\} \in \mathcal{T}$ bad!

Proposition. Let $f : X \longrightarrow Y$ be continuous.

If $x_n \in X$, $x_n \longrightarrow x$ then $f(x_n) \longrightarrow f(x)$

Idea of proof.

Expand $f(x_n) \longrightarrow f(x)$

Start: Let $V \in \mathcal{T}_Y$, $f(x) \in V$

Wish: an $N \in \mathbb{N}$ ......

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Consider this $f(x) \in V$

$$x_n \xrightarrow{\text{converges}} x \xrightarrow{f} f(x) \in V$$

$$x \in f^{-1}(V) \in \mathfrak{I}_x$$

$$\exists\, N \in \mathbb{N},\ \forall\, n \geq N \quad x_n \in f^{-1}(V) \xrightarrow{f} f(x_n) \in V$$

Remark. The converse is not true. That is,
$\exists\, f : X \longrightarrow Y$, not continuous at $x \in X$
But every $x_n \longrightarrow x$ has $f(x_n) \longrightarrow f(x)$.

Exercise. Find such example.

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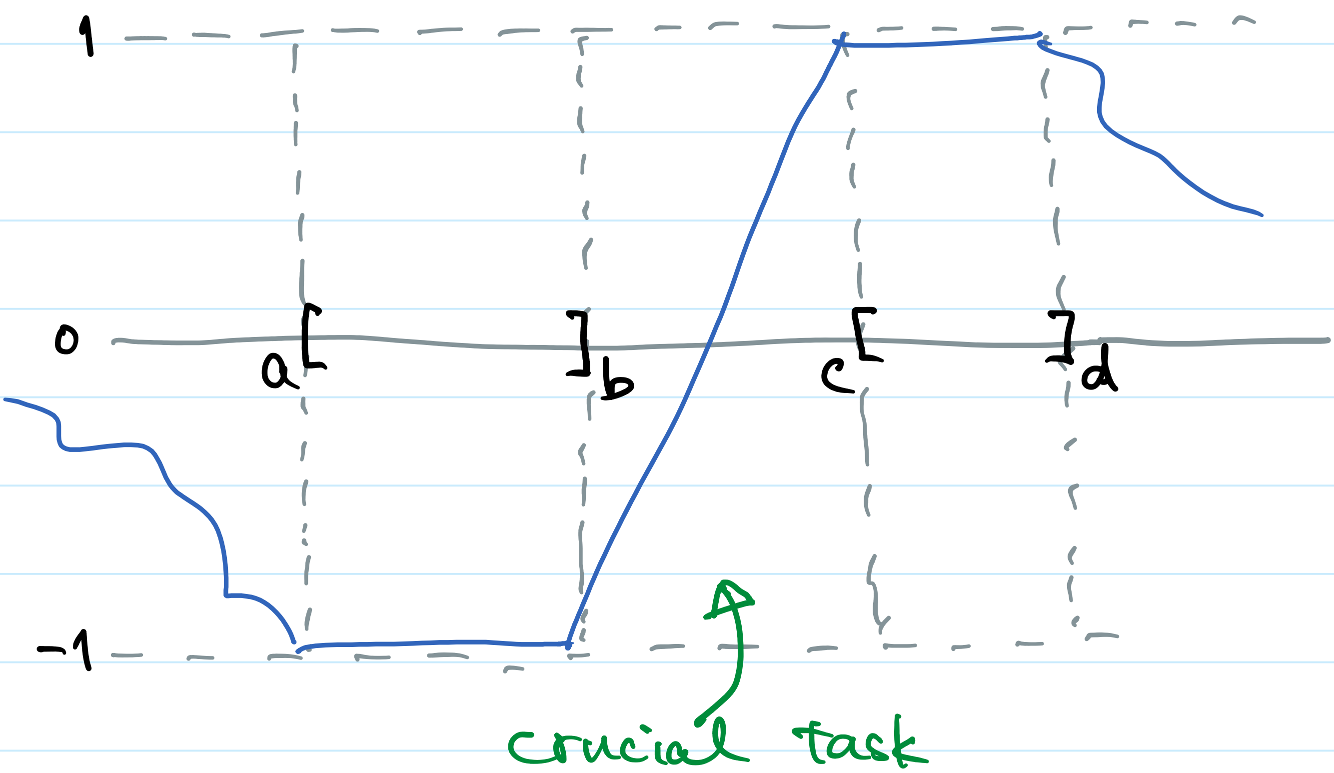
Easy Question. Given $[a,b]$, $[c,d]$, $b<c$.

Find a continuous $f:\mathbb{R}\longrightarrow[-1,1]$ such that $f|_{[a,b]}\equiv -1$, $f|_{[c,d]}\equiv 1$



$$f(x) = -1 + \frac{2}{c-b}(x-b) = \frac{2x-b-c}{c-b}, \quad x\in[b,c]$$

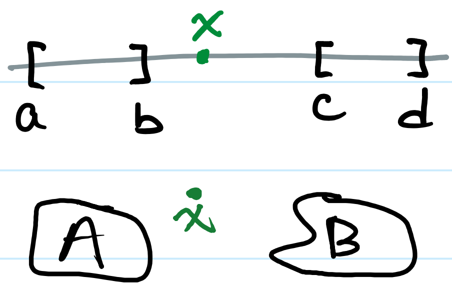
Proposition. Let (X,d) be a metric space, $A,B\subset X$ be closed and $A\cap B=\emptyset$. Then $\exists$ continuous $f:X\longrightarrow[-1,1]$ such that $f|_A\equiv -1$, $f|_B\equiv 1$.

Any insight from $A=[a,b]$, $B=[c,d]$?

Any insight from $A=[a,b]$, $B=[c,d]$?

$$f(x) = \frac{2x-b-c}{c-b} \quad \text{for } x \in [b,c]$$

$$= \frac{(x-b)-(c-x)}{(x-b)+(c-x)}$$



Idea of proof.

Define $f(x) = \dfrac{d(x,A) - d(x,B)}{d(x,A) + d(x,B)}$ $\quad x \in X$

Obviously, $f(x) = -1$ if $x \in A$

$f(x) = 1$ if $x \in B$

How to make sure that f is continuous?

(i) Fix $S \subset X$, $x \longmapsto d(x,S) : X \longrightarrow [0,\infty)$

is continuous

Why?

* Fix $z_0 \in X$, $x \longmapsto d(x,z_0)$ is continuous

Done in HW Exercise

* $d(x,S) = \inf\{d(x,z) : z \in S\}$

sup, inf of continuous functions

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(ii) Denominator $d(x,A)+d(x,B)\neq 0 \ \forall x\in X$

Why? What if denominator $=0$?

Denominator $=0 \Longrightarrow$ both $d(x,A)=0=d(x,B)$

When will $d(x,S)=0$ for $S\subset X$?

$d(x,S)=0 \Longrightarrow x\in\overline{S}$ Exercise

Denominator $=0 \Longrightarrow x\in\overline{A}=A,\ x\in\overline{B}=B$

$\therefore\ A\cap B\neq\phi$ contradiction

Urysohn Lemma (will not prove)

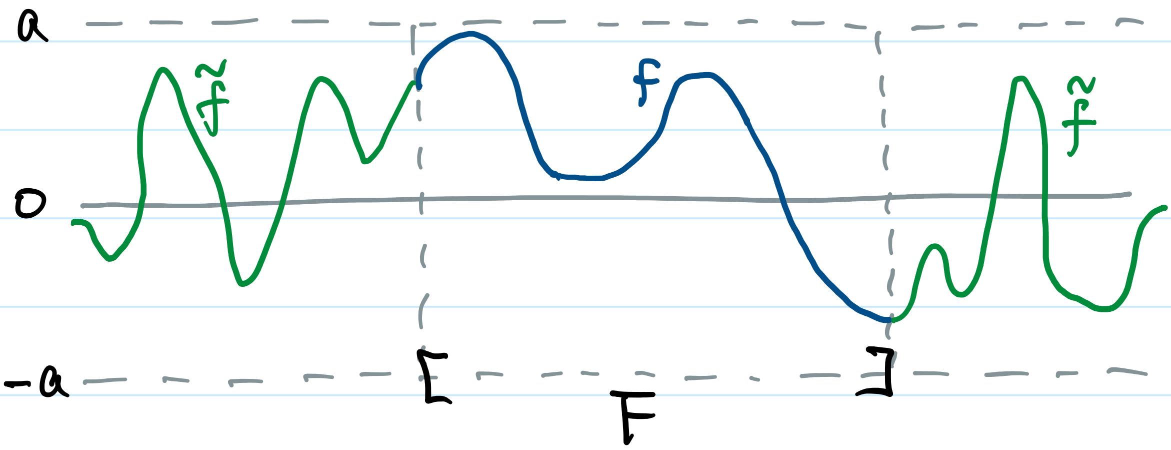
Let a space X be normal (define later), $A,B\subset X$ be closed and $A\cap B=\phi$.

Then $\exists$ continuous function $f:X\to[-1,1]$ such that $f|_A\equiv -1$, $f|_B\equiv 1$.

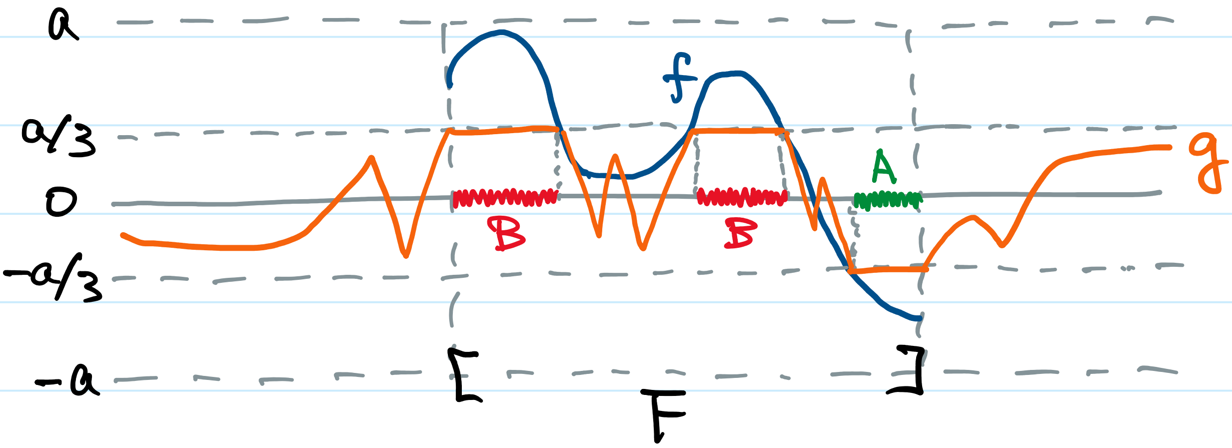
Tietz Extension Theorem. Let X be a space that the above proposition is true (i.e., X can be $\mathbb{R}$ or metric or normal).

If $F \subset X$ is closed and $f: F \longrightarrow [-a,a]$ is continuous then $\exists$ continuous extension $\tilde{f}: X \longrightarrow [-a,a]$, $\tilde{f}|_F = f$.

Picture Illustration.



Let us construct $\tilde{f}$ step by step



Let $A = f^{-1}[-a, \frac{-a}{3}]$, $B = f^{-1}[\frac{a}{3}, a]$

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By proposition, $\exists$ continuous

$g : X \longrightarrow \left[\frac{-a}{3}, \frac{a}{3}\right]$ such that

$g|_A \equiv \frac{-a}{3}$, $g|_B \equiv \frac{a}{3}$

From now on, call this g_1 ← First step

* On X, $\|g_1\| = \sup\{|g_1(x)| : x \in X\} \le \frac{a}{3}$

* On F, $\|f - g_1\| \le \frac{2a}{3}$

Second step, repeat the argument on

$f - g_1 : F \longrightarrow \left[\frac{-2a}{3}, \frac{2a}{3}\right]$ and get

$g_2 : X \longrightarrow \left[\frac{-2a}{9}, \frac{2a}{9}\right]$ and

$(f - g_1) - g_2 : F \longrightarrow \left[\frac{-4a}{9}, \frac{4a}{9}\right]$

Inductively, what do we get?

For $n=1,2,3,\cdots$

$$g_n : X \longrightarrow \left[\frac{-a}{3}\left(\frac{2}{3}\right)^{n-1}, \frac{a}{3}\left(\frac{2}{3}\right)^{n-1}\right]$$

$$f - \sum_{k=1}^{n} g_k : F \longrightarrow \left[-a\left(\frac{2}{3}\right)^{n}, a\left(\frac{2}{3}\right)^{n}\right]$$

By standard theory in analysis (which?)

$$\sum_{k=1}^{n} g_k \xrightarrow[\text{uniformly}]{} \tilde{f} : X \longrightarrow [-a,a]$$

Moreover, what happens to

$$\| f - \tilde{f} \|_F$$

$$\leq \| f - \sum_{k=1}^{n} g_k \|_F + \| \sum_{k=1}^{n} g_k - \tilde{f} \|_X$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \forall \varepsilon > 0$$

$$\therefore \tilde{f}|_F \equiv f.$$