Notes on the Extreme Value Theorem (Flowchart)

Statement of the EVT

Assumptions: (i) f defined on $[a, b]$. (ii) f continuous.
Claim: f has 'absolute' (or 'global') maximum value, M, attained by the 'absolute' maximum point(s) x_M. Similarly, f has 'absolute' minimum value m attained by the 'absolute' minimum point(s) x_m (they are both in $[a, b]$!)

Remark. When we write 'point(s)', we want to emphasize the fact that there may be more than one point.

Flowchart of the Proof.

Look at the 'range', i.e. R_f, of the function f defined on the domain $[a, b]$.

$\downarrow$

Show, via contradiction proof, that R_f is a bounded set. (Here we have to use the 'Deep Result' of real numbers, the Bolzano-Weierstrass Theorem and also the continuity of the function f.)

$\downarrow$

By subtracting $\frac{1}{n}$ from $\sup(R_f)$, we get a sequence of points (y_n) s.t. $\lim_{n\to\infty} y_n = \sup(R_f)$. $\boxed{\text{This sequence is a subset of } R_f \text{ !}}$

$\downarrow$

Note that, by definition of 'range', each y_n is the image of at least one x_n via the function f. Hence we get a sequence (x_n) in $[a, b]$

$\downarrow$

Note that while y_n go to $\sup(R_f)$, the sequence (x_n) may not be convergent in $[a, b]$. $\boxed{\text{I.e. it may not have a limit in the set } [a, b]}$ Apply Bolzano-Weierstrass Theorem here to get a subsequence (x_{n_k}) which is convergent in $[a, b]$.

$\downarrow$

Denote the limit of (x_{n_k}) by α, then by continuity of f, we get $\lim_{k\to\infty} x_{n_k} = \alpha$ and can show that α is the absolute max. point.

In the following pages, we will explain the above 'program' in detail.

$\star$ We will only consider the 'absolute maximum' case. The proof of the 'absolute minimum' case is completely analogous.

Notes on the Extreme Value Thm.

Tools needed: Bolzano - Weierstraß Thm., i.e.

Every bounded sequence has a convergent subsequence.

(E.V.T.)

Q:) How to use this to show the Extreme Value thm.?

E.V.T.

Statement of the EVT:

Assump. $f : [a,b] \longrightarrow \mathbb{R}$ is cont. $\Leftarrow$ Two important points here! (I) $[a,b]$ — closed & bounded (II) f — cont.

Conclusion: $\exists$ absolute maximum M & absolute minimum m.

More precisely, there exist m, M and 2 points x_m & x_M in the domain $[a,b]$ s.t.

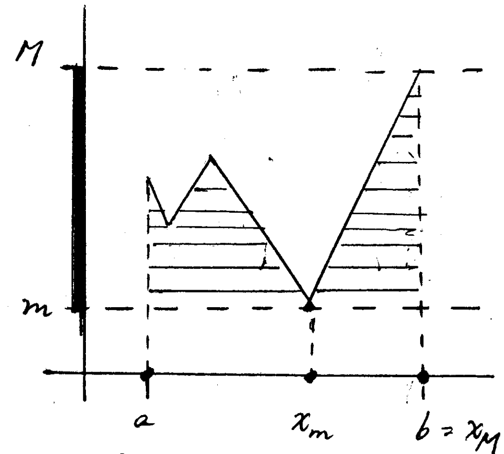
$$f(x_m) = m \quad \& \quad f(x_M) = M.$$

Terminologies: x_m — absolute min. pt. $\quad m$ — abs. min. (value)

x_M — " max. " $\quad M$ — abs. max. (value).

Proof of the EVT.

First, consider the picture:



Q: What do we see from this picture?

Q: More precisely, what do we see on the y-axis?

Q: " " " " " " " " " related to the function f?

Answer: We see the set ("range of f") denoted by

$$R_f = \{ y : y = f(x) \ \exists x \in [a,b] \}$$

(and given by; for some)

We also see that the set R_f has a supremum (denoted by $\sup(R_f)$)
and an infimum (" " $\inf(R_f)$).

Q: Why?

A: To show that R_f has supremum & infimum, we use the Deep Result of real nos. which says. (note that we had a similar statement for bounded sequences before!)

$$\boxed{\text{Every bounded set } S \text{ in the set of real nos. (i.e. } \mathbb{R}) \text{ has a supremum and an infimum.}} \qquad (*)$$

To use (*), we need to show that

R_f (which is a subset of $\mathbb{R}$!)

is bounded from above ——— ①

& " from below! ——— ②

(In the following, we'll show ①, ② is similar!)

Claim: R_f is bounded from above.

Pf: By contradiction.

Suppose R_f is not bounded from above, then

$$\sim (\exists K (\forall x \in [a,b] (f(x) \leq K)))$$

$$\Leftrightarrow \quad \forall K (\exists x \in [a,b] (f(x) > K)))$$

depends on K!

I'm using ( ) instead of : to make it clear what the "scope" of each quantifier is!

For convenience, we change notation & write "n" instead of "K" to get (& we specialize K to "natural nos."!)

$$\Leftrightarrow \quad \forall n (\exists x_n \in [a,b] (f(x_n) > n))) \qquad \text{———} \quad (**)$$

(We write "x sub n" to remind ourselves that x depends on n!)

Concl:. We have obtained a bounded sequence (x_n) in $[a,b]$.

Now we apply the Bolzano-Weierstraß Thm to obtain a convergent subsequence (x_{n_k}) of the sequence (x_n)

Let the limit of (x_{n_k}), as $k \to \infty$, be c. $(c \in [a,b])$

Then $\lim_{k\to\infty} x_{n_k} = c.$ ——— ③

Since f is a cont. fn. on $[a,b]$, & $c \in [a,b]$, we obtain

$$\lim_{k\to\infty} f(x_{n_k}) = f(c) \quad \text{——— ④}$$

Next, we look at ④, thinking back at (*) on p. 2., to see what we have actually proven (up to this point!)

(*) says $f(x_n) > n \quad \forall n$

$\Rightarrow \quad f(x_{n_k}) > n_k \quad \forall n_k$ ——— ⑤

Applying ⑤ to ④, we obtain

$$f(c) = \lim_{k\to\infty} f(x_{n_k}) > \lim_{k\to\infty} n_k \quad \forall n_k \text{ nat. no.}$$

The right-hand side of this inequality goes to $+\infty$, while the left-hand side is a finite no. Therefore we arrived at a contradiction.

Concl: The set R_f is bounded from above. $\square$

Next step.

$\boxed{\text{Claim:}}$ $\exists x_m, x_M \ (f(x_m)=m,\ f(x_M)=M)$.

Pf: (I) By the previous step.

R_f is <u>bounded above</u> & <u>bounded below</u>

$$\Rightarrow \quad \exists \sup(R_f) \qquad \text{—— (6)}$$
$$\inf(R_f). \qquad \text{—— (7)}$$

$\forall$ ~~[illegible]~~

$$\forall y \,(y \in R_f \Rightarrow y \leq \sup(R_f))$$

i.e. $\sup(R_f)$ is an upper bound for R_f

$$\& \quad \forall \varepsilon \,(\varepsilon > 0 \Rightarrow \exists y \,(y \in R_f \ \& \ y > \sup(R_f) - \varepsilon)). \qquad \text{—— (8)}$$

i.e. Anything less than $\sup(R_f)$ isn't an upper bdd. for R_f!

Let's now use (8) to generate a sequence of nos.

Let $\varepsilon = \frac{1}{n}$, $n=1,2,3,\cdots$, then (8) gives

$$\forall n \,(n \in \mathbb{N} \Rightarrow \exists y_n \,(y_n \in R_f \ \& \ y_n > \sup(R_f) - \varepsilon))$$

$$\Leftrightarrow \quad \forall n \,(n \in \mathbb{N} \Rightarrow \exists y_n \,(\exists x_n \,(f(x_n) = y_n) \ \& \ \underset{\underset{y_n}{\|}}{f(x_n)} > \sup(R_f) - \tfrac{1}{n})).$$

Here we obtain a sequence (y_n) (in the range of f) s.t. $\lim_{n\to\infty} y_n = \sup(R_f)$.

<u>Next</u> we have ($\because$ $y_n \leq \sup(R_f)$ by definition of "supremum"!)

$$\sup(R_f) - \tfrac{1}{n} < y_n \leq \sup(R_f) < \sup(R_f) + \tfrac{1}{n}$$

this is simple!

Concl: We have obtained a convergent sequence $(y_n) \subseteq R_f$.

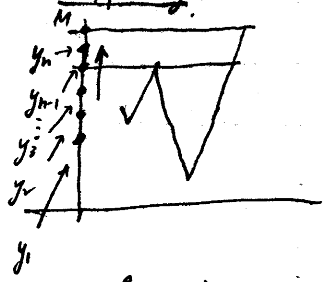
& $\lim_{n\to\infty} y_n = \sup(R_f)$. $\square$

(II) Now we let $M = \sup(R_f)$.

Claim:- $\exists x_M \in [a,b]$ s.t. $f(x_M) = M$. ($= \sup(R_f)$).

Pf: Before proceeding to the proof, let's mention one possible difficulty.

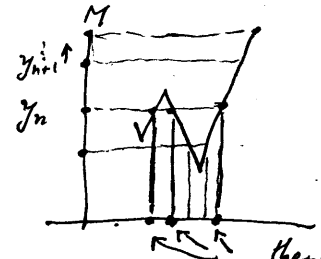
We know (i) $\lim_{n\to\infty} y_n = M$ ($= \sup(R_f)$).
(see fig (ii))



(fig (ii)).

(ii) $\forall y_n \left(\exists x_n \left(f(x_n) = y_n\right)\right)$.

(see fig (iii)).



(fig (iii))

there are more than 1 choice for x_n s.t. $f(x_n) = y_n$.

therefore, from the sequence (y_n), we obtain the sequence (x_n)

Difficulty! (iii) ** While the sequence (y_n) is convergent, the sequence (x_n) may not be convergent!

Rk: It's a good exercise to find an example for this!

(Proof cont'd) From (y_n), we obtain (x_n) s.t. $f(x_n) = y_n$.

$(x_n) \subseteq [a,b]$ therefore is bounded above by b.

By Bolzano-Weierstraß Thm, $\exists (n_k) \subseteq (n)$ s.t. (x_{n_k}) is convergent

Let $\lim_{k\to\infty} x_{n_k} = \alpha$

Then by continuity of f on $[a,b]$,

$$\underbrace{\lim_{k\to\infty} f(x_{n_k})}_{\substack{\| \\ f(\alpha)}} = \lim_{k\to\infty} y_{n_k} = \lim_{n\to\infty} y_n = \sup(R_f) = M$$

& $f(\alpha) \Rightarrow f(\alpha) = M$, $\therefore$ α is the absolute max pt. x_M. $\square$