§65 Negative Powers of $(z-z_0)$

eg2 For $0<|z|<\infty$, we have $0<\left|\frac{1}{z}\right|<\infty$.

$$\Rightarrow \cosh\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \cdot \frac{1}{z^{2n}} \quad \left(0<\frac{1}{|z|}<\infty\right)$$

$$\left(0<|z|<\infty\right)$$

$$= 1 + \frac{1}{2!\,z^2} + \frac{1}{4!\,z^4} + \cdots \quad (0<|z|<\infty)$$

eg3 : $f(z) = \dfrac{1+2z^2}{z^3+z^5}$ in powers of z $(0<|z|<1)$

Soln : $f(z) = \dfrac{1+2z^2}{z^3(1+z^2)} = \dfrac{1}{z^3}\,\dfrac{2(1+z^2)-1}{1+z^2}$

$$= \frac{1}{z^3}\left[2 - \frac{1}{1+z^2}\right] \quad \left(\begin{array}{c} 0<|z|<1 \\ \Downarrow \\ 0<|z|^2<1 \end{array}\right)$$

$$= \frac{1}{z^3}\left[2 - (1 - z^2 + z^4 - \cdots)\right] \quad (0<|z|<1)$$

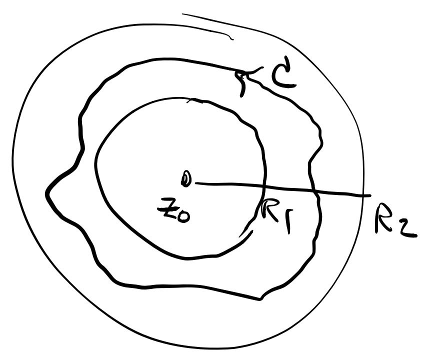
$$= \frac{1}{z^3}\left[1 + z^2 - z^4 + z^6 - \cdots\right]$$

$$= \frac{1}{z^3} + \frac{1}{z} - z + z^3 - \cdots \quad (0<|z|<1)$$

(eg 1 & eg 4 : Reading ex!)

§66 Laurent Series

Thm (Laurent) Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in the domain. Then at each point in the domain, $f(z)$ has the series representation



$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad (R_1 < |z - z_0| < R_2)$$

where

$$\begin{cases} a_n = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(s)\,ds}{(s - z_0)^{n+1}}, & n = 0, 1, 2, \dots \\ \\ b_n = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(s)\,ds}{(s - z_0)^{-n+1}}, & n = 1, 2, 3, \dots \end{cases}$$

Let $c_n = \begin{cases} a_n & \text{if } n \geq 0 \\ b_{-n} & \text{if } n < 0 \end{cases}$

Then $\quad f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \qquad (R_1 < |z-z_0| < R_2)$

where $\quad c_n = \frac{1}{2\pi i} \int_C \frac{f(s)\,ds}{(s-z_0)^{n+1}}, \quad \forall n = 0, \pm 1, \pm 2, \dots$

Note: (i) Both forms are called a Laurent Series expansion (or representation) of $f(z)$

(ii) The Theorem asserts that both series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \& \quad \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad \text{converge}$$

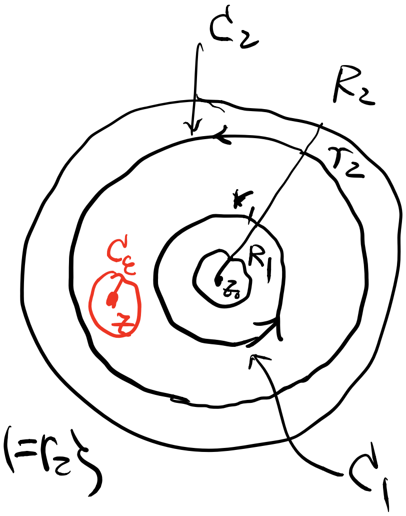
for $z \in \{ R_1 < |z-z_0| < R_2 \}$ and their sum is $f(z)$.

(iii) R_1 could be zero; R_2 could be infinite:

we may have
$$\begin{cases} 0 < R_1 < |z-z_0| < R_2 < \infty \\ 0 < |z-z_0| < R_2 < \infty \\ 0 < R_1 < |z-z_0| < \infty \\ 0 < |z-z_0| < \infty \end{cases}$$

(iv) In case that f is actually analytic in $|z-z_0| < R_2$. Then it is easy to show that $b_n = 0$, $\forall n = 1, 2, 3, \dots$ and the Laurent series becomes the Taylor series about z_0. (Ex!)

§67 Proof of Laurent's Theorem



Case $z_0 = 0$

Consider $\{r_1 \le |z| \le r_2\}$ with
$R_1 < r_1 < r_2 < R_2$.

Let $C_1 : \{|z| = r_1\}$ & $C_2 : \{|z| = r_2\}$

Then f is analytic on C_1 & C_2, and between them.

Let $z \in \{r_1 < |z| < r_2\}$. Then $\exists\, \varepsilon > 0$ s.t.

$B_\varepsilon(z) \subset \{r_1 < |z| < r_2\}$ with bdy C_ε.

Applying Cauchy-Goursat Thm to the analytic function $\frac{f(s)}{s-z}$, we have

$$\int_{C_2} \frac{f(s)\,ds}{s-z} - \int_{C_1} \frac{f(s)\,ds}{s-z} - \int_{C_\varepsilon} \frac{f(s)\,ds}{s-z} = 0$$

Then Cauchy Integral Formula

$$f(z) = \frac{1}{2\pi i} \int_{C_\varepsilon} \frac{f(s)\,ds}{s-z} = \frac{1}{2\pi i}\left[\int_{C_2} \frac{f(s)\,ds}{s-z} - \int_{C_1} \frac{f(s)\,ds}{s-z}\right]$$

For s on C_2, $\left|\frac{z}{s}\right| = \frac{|z|}{r_2} < 1$, we have

$$\frac{1}{2\pi i}\int_{C_2} \frac{f(s)\,ds}{s-z} = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s}\left(\frac{1}{1-\frac{z}{s}}\right) ds$$

$$= \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s}\left[\sum_{n=0}^{N-1}\left(\frac{z}{s}\right)^n + \frac{\left(\frac{z}{s}\right)^N}{1-\frac{z}{s}}\right] ds$$

$$= \sum_{n=0}^{N-1}\left(\frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s^{n+1}}\,ds\right) z^n$$

$$+ \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\left(\frac{z}{s}\right)^N ds.$$

As in the proof of Taylor expansion,

$$\left|\frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\left(\frac{z}{s}\right)^N ds\right| \to 0 \quad \text{as } N \to \infty$$

$$\therefore \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\,ds = \sum_{n=0}^{\infty} a_n z^n,$$

where $a_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)\,ds}{s^{n+1}} = \frac{1}{2\pi i}\int_{C} \frac{f(s)}{s^{n+1}}\,ds$

by principle of deformation of paths.

On the other hand, for $s \in C_1$, $\left|\frac{z}{s}\right| = \frac{|z|}{r_1} > 1$.

Hence

$$\frac{-1}{2\pi i}\int_{C_1}\frac{f(s)\,ds}{s-z} = \frac{1}{2\pi i}\int_{C_1}\frac{f(s)\,ds}{z\left(1-\frac{s}{z}\right)} \qquad \left(\begin{array}{c}\text{Note}: |z|>r_1>0 \\ \Rightarrow z\neq 0\end{array}\right)$$

$$= \frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{z}\left[\sum_{n=0}^{N-1}\left(\frac{s}{z}\right)^n + \frac{\left(\frac{s}{z}\right)^N}{1-\frac{s}{z}}\right]ds$$

$$= \sum_{\substack{\cancel{n=0}\\ k=1}}^{\cancel{N-1}\;N}\left(\frac{1}{2\pi i}\int_{C_1} f(s)\,s^{\cancel{n}\;k-1}\,ds\right)\frac{1}{z^{\cancel{n+1}\;k}} \qquad (k=n+1)$$

$$+\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{z-s}\left(\frac{s}{z}\right)^N ds$$

$$= \sum_{n=1}^{N}\left(\frac{1}{2\pi i}\int_{C_1} f(s)\,s^{n-1}\,ds\right)\frac{1}{z^n}$$

$$+\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{z-s}\left(\frac{s}{z}\right)^N ds$$

$$= \sum_{n=1}^{N}\left(\frac{1}{2\pi i}\int_{C_1}\frac{f(s)\,ds}{s^{-n+1}}\right)\frac{1}{z^n} + \frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{z-s}\left(\frac{s}{z}\right)^N ds$$

Now $\left|\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{z-s}\left(\frac{s}{z}\right)^N ds\right|$

$$\le \frac{1}{2\pi}\,\frac{M_1}{r-r_1}\left(\frac{r_1}{r}\right)^N\cdot 2\pi r_1$$

$$= \frac{M_1 r}{r-r_1}\left(\frac{r_1}{r}\right)^N \longrightarrow 0 \quad \text{as } N\to+\infty$$

$\left(\text{where} \quad M_1 = \sup_{|s|=r_1} |f(s)| \quad , \quad r = |z| > r_1 \right)$

$$\therefore \quad -\frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{s-z}\,ds = \sum_{n=1}^{\infty} \frac{b_n}{z^n}$$

where $\displaystyle b_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{s^{-n+1}}\,ds = \frac{1}{2\pi i}\int_{C} \frac{f(s)\,ds}{s^{-n+1}}$

by principle of defamation of paths.

Since $R_1 < r_1 < |z| < r_2 < R_2$ are arbitrary, we have proved the case for $z_0 = 0$.

General case $z_0 \neq 0$ follows easily. #

§68 Examples

eg1: Find Laurent Series expansion of $f(z) = \dfrac{1}{z(1+z^2)}$ on $0<|z|<1$. $\left(\begin{array}{l}\text{Note: } f \text{ is analytic in } 0<|z|<1 \\ \text{because singularities are } z=0, \pm i\end{array}\right)$

Soln: $$f(z) = \frac{1}{z(1+z^2)}$$

$$= \frac{1}{z}\left(\frac{1}{1-(-z^2)}\right) = \frac{1}{z}\left(1 + (-z^2) + (-z^2)^4 + \cdots\right)$$

$$= \frac{1}{z}\sum_{n=0}^{\infty} (-z^2)^n \qquad \left(\begin{array}{l} 0<|z|<1 \\ \Rightarrow |-z^2|<1 \end{array}\right)$$

$$= \frac{1}{z}\sum_{n=0}^{\infty} (-1)^n z^{2n}$$

$$= \sum_{n=0}^{\infty} (-1)^n z^{2n-1}$$

$$= \frac{1}{z} - z + z^3 - z^5 + \cdots$$

is the Laurent Series of f about $z_0 = 0$.

$$\left(\text{Or: } f(z) = \left(\sum_{n=0}^{\infty} (-1)^{n+1} z^{2n+1} \right) + \frac{1}{z}. \right)$$

eg2: $f(z) = \frac{z+1}{z-1}$ analytic in $D_1 = \{ |z| < 1 \}$

& $D_2 = \{ 1 < |z| < \infty \}$

On $D_1 = \{ |z| < 1 \}$, $f(z) = \frac{z+1}{z-1}$ has a Taylor expansion

$$= -1 - 2 \sum_{n=1}^{\infty} z^n \quad (|z| < 1) \quad \underline{\text{Ex!}}$$

On $D_2 = \{ 1 < |z| < \infty \}$, we have

$$f(z) = \frac{z+1}{z-1} = \frac{z+1}{z} \cdot \frac{1}{1 - \left(\frac{1}{z}\right)}$$

$$= \left(1 + \frac{1}{z}\right)\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots \right)$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots$$

$$+ \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$$= 1 + \frac{2}{z} + \frac{2}{z^2} + \frac{2}{z^3} + \cdots \quad (1 < |z| < \infty)$$

$$= 1 + \sum_{n=1}^{\infty} \frac{2}{z^n} \qquad (1 < |z| < \infty)$$

$$f(z) = \left(1 + \frac{1}{z}\right)\left(\sum_{n=0}^{\infty} \frac{1}{z^n}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

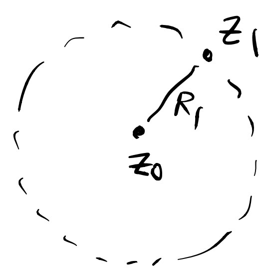
$$= 1 + \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{\cancel{k}\,n=1}^{\infty} \frac{1}{z^{\cancel{k}\,n}}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{2}{z^n}$$

(Egs 3&4 : Reading Ex!)

§69 Absolute and Uniform Convergence of Power Series

Thm 1 If a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges when $z = z_1$ $(z_1 \neq z_0)$, then it is absolutely convergent at each point z in the open disk $|z - z_0| < R_1$ where $R_1 = |z_1 - z_0|$.



Pf: $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ converges

$\Rightarrow \lim_{n\to\infty} a_n (z_1 - z_0)^n = 0$

$\Rightarrow \{a_n (z_1 - z_0)^n\}$ bounded.

$\Rightarrow \exists M > 0$ s.t. $|a_n(z_1-z_0)^n| \le M, \forall n = 0,1,2,\cdots$

Hence $\forall z \in \{|z-z_0| < R_1 = |z_1 - z_0|\}$, we have

$$|a_n(z-z_0)^n| = |a_n(z_1-z_0)^n| \left|\frac{z-z_0}{z_1-z_0}\right|^n$$

$$\le M\rho^n \quad , \text{where } \rho = \frac{|z-z_0|}{|z_1-z_0|} = \frac{|z-z_0|}{R_1} < 1.$$

$\therefore \sum_{n=0}^{\infty} |a_n(z-z_0)^n| \le \sum_{n=0}^{\infty} M\rho^n$ converges by comparison test (since $0<\rho<1$)

\#

Def: The <u>greatest circle</u> centered at z_0 such that the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges at each point inside is called the <u>circle of convergence</u> of the series.

<u>Cor</u>: For any z_2 outside the circle of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, the series $\sum_{n=0}^{\infty} a_n(z_2-z_0)^n$ <u>diverges</u>.

Pf: If $\sum_{n=0}^{\infty} a_n(z_2-z_0)^n$ converges, then by Thm 1, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely in $|z-z_0| < |z_2-z_0|$

$\Rightarrow z_2$ on (or inside) the circle of convergence.

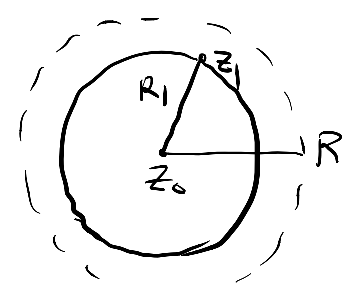
Contradiction. \#

<u>Def</u>: Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges $\forall z$ in a region Ω with sum $S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$. If $\forall \varepsilon > 0$, $\exists N_\varepsilon > 0$ independent of $z \in \Omega$ such that $\forall z \in \Omega$,

$$|\rho_N(z)| = |S(z) - S_N(z)| < \varepsilon, \ \forall N > N_\varepsilon.$$

Then $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is said to <u>uniformly convergent</u> in the region Ω. (or the convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is <u>uniform</u> in the region Ω), where $S_N(z)$ is the Nth partial sum & $\rho_N(z)$ is the remainder.

<u>Thm 2</u> If z_1 is a point insider the circle of convergence $|z-z_0|=R$ of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, then that series must be uniformly convergent in the closed disk $|z-z_0| \leq R_1 = |z_1 - z_0|$.



<u>Pf</u>: By Thm 1, $\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$ is absolutely convergence.

$\Rightarrow \sum_{n=0}^{\infty} |a_n(z_1-z_0)^n|$ converges.

$\Rightarrow \forall \varepsilon > 0$, $\exists N_\varepsilon$ (indep. of z on $|z-z_0| \leq R_1$)

s.t. $\sum_{n=N}^{\infty} |a_n(z_1-z_0)^n| < \varepsilon$, $\forall N > N_\varepsilon$.

Then $\quad |\rho_N(z)| = \left|\sum_{n=N}^{\infty} a_n (z-z_0)^n\right| \le \sum_{n=N}^{\infty} |a_n||z-z_0|^n$

$$\le \sum_{n=N}^{\infty} |a_n||z_1-z_0|^n < \varepsilon, \quad \forall N > N_\varepsilon$$

$\therefore \quad \sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges uniformly in $|z-z_0| \le R_1 = |z_1-z_0|$.

$\#$