§38 Zeros & Singularities of trigonometric functions

Def: A <u>zero</u> of a given function f is a cpx number z_0 s.t. $f(z_0)=0$.

Thm: The zeros of $\sin z$ & $\cos z$ in $\mathbb{C}$ are the same as the zeros of $\sin x$ and $\cos x$ on $\mathbb{R}$.

That is
$$\begin{cases} \sin z = 0 \iff z = n\pi & (n\in\mathbb{Z}) \\ \cos z = 0 \iff z = \frac{\pi}{2} + n\pi & (n\in\mathbb{Z}) \end{cases}$$

Pf: By property (7) of previous section:

$$0 = |\sin z|^2 = \sin^2 x + \sinh^2 y$$

$$\iff \sin x = 0 \quad \& \quad \underbrace{\sinh y}_{\frac{e^y-e^{-y}}{2}} = 0 \quad (y\in\mathbb{R})$$

$$\iff x = n\pi,\ n\in\mathbb{Z} \quad \& \quad y=0$$

$$\iff z = n\pi,\ n\in\mathbb{Z}.$$

Similarly for $\cos z$. #

Def (Other trigonometric functions)

$$\tan z \overset{\text{def}}{=} \frac{\sin z}{\cos z}, \quad \cot z \overset{\text{def}}{=} \frac{\cos z}{\sin z}$$

$\sec z \overset{\text{def}}{=\!=} \dfrac{1}{\cos z}$, $\csc z \overset{\text{def}}{=\!=} \dfrac{1}{\sin z}$

Note: By $\overset{\text{above}}{\smile}$ Thm,

(i) $\tan z$, $\sec z$ are analytic except at the singularities $z = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$

(ii) $\cot z$, $\csc z$ are analytic except at the singularities $z = n\pi$, $n \in \mathbb{Z}$.

Prop:
$$\begin{cases} \dfrac{d}{dz}\tan z = \sec^2 z \\ \dfrac{d}{dz}\cot z = -\csc^2 z \\ \dfrac{d}{dz}\sec z = \sec z \tan z \\ \dfrac{d}{dz}\csc z = -\csc z \cot z \end{cases}$$
(Ex!)

§39 Hyperbolic Functions

Recall for $y \in \mathbb{R}$,
$$\begin{cases} \sinh y = \dfrac{e^y - e^{-y}}{2} \\ \cosh y = \dfrac{e^y + e^{-y}}{2} \end{cases}$$
(no i)

Hence we define

Def: $\forall z \in \mathbb{C}$, $\sinh z = \dfrac{e^z - e^{-z}}{2}$ (No i)

$$\cosh z = \frac{e^{z}+e^{-z}}{2}$$

the <u>hyperbolic sine</u> and <u>hyperbolic cosine</u> functions for cpx numbers.

<u>Properties</u> :

(1) $\begin{cases} \sinh(iz) = i\sin z \ , & \cosh(iz) = \cos z \\ \sin(iz) = i\sinh z \ , & \cos(iz) = \cosh z \end{cases}$

(2) $\begin{cases} \sinh(-z) = -\sinh z & \text{odd} \\ \cosh(-z) = \cosh z & \text{even} \end{cases}$

(3) $\cosh^2 z - \sinh^2 z = 1$

(4) $\begin{cases} \sinh(z_1+z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2 \\ \cosh(z_1+z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2 \end{cases}$

(5) $\begin{cases} \sinh z = \sinh x \cos y + i \cosh x \sin y \\ \cosh z = \cosh x \cos y + i \sinh x \sin y \end{cases}$

(6) $\begin{cases} |\sinh z|^2 = \sinh^2 x + \sin^2 y \\ |\cosh z|^2 = \sinh^2 x + \cos^2 y \end{cases}$ (unbounded in x-direction)

(7) $\begin{cases} \sinh z = 0 \iff z = n\pi i \ , & n \in \mathbb{Z} \\ \cosh z = 0 \iff z = \left(\frac{\pi}{2} + n\pi\right) i \ , & n \in \mathbb{Z} \end{cases}$

(Ex!)

Def: (Other hyperbolic functions)

$$\tanh z \overset{\text{def}}{=} \frac{\sinh z}{\cosh z} \quad (\text{hyperbolic tangent})$$

$$\coth z \overset{\text{def}}{=} \frac{\cosh z}{\sinh z} \quad (\quad " \quad \text{cotangent})$$

$$\operatorname{sech} z \overset{\text{def}}{=} \frac{1}{\cosh z} \quad (\quad " \quad \cdots \quad)$$

$$\operatorname{csch} z \overset{\text{def}}{=} \frac{1}{\sinh z} \quad (\quad " \quad \cdots \quad)$$

Prop:

$$\begin{cases} \frac{d}{dz}\sinh z = \cosh z \quad \frac{d}{dz}\cosh z = \sinh z \\ \frac{d}{dz}\tanh z = \operatorname{sech}^2 z \quad \frac{d}{dz}\coth z = -\operatorname{csch}^2 z \\ \frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z \tanh z \\ \frac{d}{dz}\operatorname{csch} z = -\operatorname{csch} z \coth z \end{cases}$$

(Pf: Ex)

§31 The Logarithmic function

Def: The (multiple-valued) logarithmic function of $z = re^{i\theta} \in \mathbb{C} \setminus \{0\}$ is

$$\log z \overset{\text{def}}{=} \ln r + i \arg z$$
$$= \ln r + i(\theta + 2n\pi), \quad n = 0, \pm 1, \pm 2, \cdots$$

where $\ln$ is the natural log on $\mathbb{R}^+$.

Notes: (1) $\log z$ is a set and can be written as

$$\boxed{\log z = \ln|z| + i\arg z}$$

sets

(It means: $w \in \log z \Leftrightarrow w$ can be written in the form $\ln|z| + i\theta$ for some $\theta \in \arg z$)

(2) $\forall\, w \in \log z$, we have

$$e^w = e^{\ln|z| + i\theta} \quad \text{for some } \theta \in \arg z$$

$$= e^{\ln|z|} e^{i\theta} = |z|(\cos\theta + i\sin\theta)$$

$$= z$$

$\therefore$ $\log z$ is the "inverse" of $\exp z$!

(3) However

$$\log e^z = \log(e^{x+iy}) = \ln e^x + i(y + 2n\pi), \quad n \in \mathbb{Z}$$

$$= x + i(y + 2n\pi), \quad n \in \mathbb{Z}$$

$$= z + 2n\pi i, \quad n \in \mathbb{Z}.$$

Def: Principal value of $\log z$, denoted by $\operatorname{Log} z$

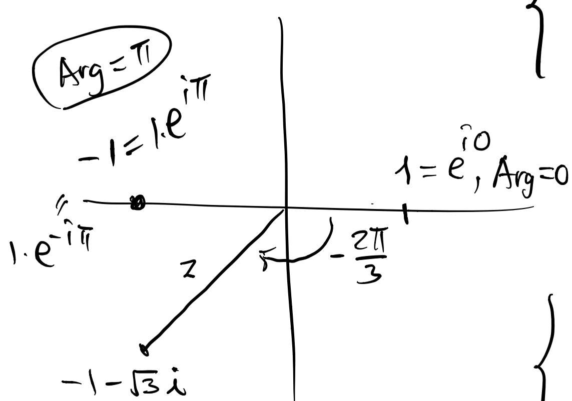
is $\boxed{\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z} \in \mathbb{C}$

principal argument of z $\in (-\pi, \pi]$

Note: Hence $\log z = \operatorname{Log} z + 2n\pi i$, $n \in \mathbb{Z}$.

§32 Examples

egs 1, 2 & 3



$\parallel$

$2e^{-\frac{2\pi}{3}i}$

$\operatorname{Arg} = -\frac{2\pi}{3}$

$$\begin{cases} \operatorname{Log}(-1-\sqrt{3}i) = \ln 2 - \frac{2\pi}{3} i \\ \log(-1-\sqrt{3}i) = \ln 2 - \frac{2\pi}{3} i + 2n\pi i \end{cases}$$

$$= \ln 2 + (2n - \tfrac{2}{3})\pi i \quad n \in \mathbb{Z}$$

$$\begin{cases} \operatorname{Log} 1 = 0 \\ \log 1 = 0 + 2n\pi i \end{cases}$$

$$= 2n\pi i, \quad n \in \mathbb{Z}$$

$$\begin{cases} \operatorname{Log}(-1) = \pi i \\ \log(-1) = (2n+1)\pi i, \quad n \in \mathbb{Z} \end{cases}$$

eg: $\exp\left(\frac{1}{n} \log z\right) = \exp\left[\frac{1}{n}\left[\ln|z| + i(\Theta + 2k\pi)\right]\right\}$ $k \in \mathbb{Z}$

$$= \sqrt[n]{|z|} \exp\left[i\left(\frac{\Theta}{n} + \frac{2k\pi}{n} \right) \right], \quad k \in 0, 1, 2, \cdots, n-1. \qquad (\Theta = \operatorname{Arg} z)$$

gives the n-roots of z.