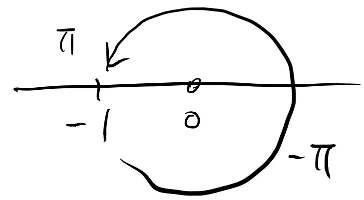
Egs (1) $z=-1$, then $\underline{\mathrm{Arg}(-1)} = \pi$ (not $-\pi$)



$(-\pi, \pi]$

$\arg(-1) = \{ \cdots, \pi - 2\pi, \pi, \pi + 2\pi, \pi + 4\pi, \cdots \}$

$= \pi + 2k\pi, \ \forall k \in \mathbb{Z}$

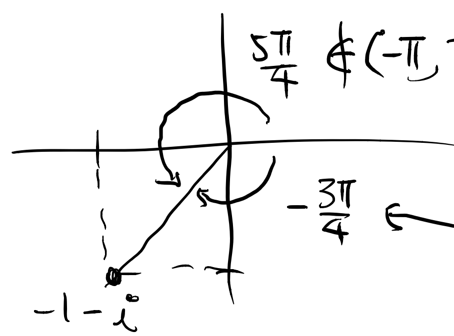
($\leftarrow \mathrm{Arg}(-1)$)

$= -\pi + 2k\pi, \ \forall k \in \mathbb{Z}$ (Ex!)

(an argument, but not principal)

k are different here, just an index.

(2) $z = -1 - i$



$\frac{5\pi}{4} \notin (-\pi, \pi]$ argument but principal

$-\frac{3\pi}{4} \leftarrow \mathrm{Arg}(-1-i)$

$\arg z = \{ -\frac{3\pi}{4} + 2k\pi : k \in \mathbb{Z} \}$

$= \{ \frac{5\pi}{4} + 2k\pi : k \in \mathbb{Z} \}$

$= \{ \cdots, -\frac{3\pi}{4}, \frac{5\pi}{4}, \frac{13\pi}{4}, \cdots \}$

In fact, $\boxed{\arg z = \text{any value of argument} + 2k\pi \quad k = 0, \pm 1, \pm 2, \cdots}$

<u>Notation</u>:

Define $\underline{e^{i\theta} = \cos\theta + i\sin\theta}$, $\forall \theta \in \mathbb{R}$

(Euler formula)

Then $\underline{z = r(\cos\theta + i\sin\theta) = |z| e^{i\theta}}$

is the <u>exponential form</u> of z.

<u>eg3</u>: $z = -1-i$, $\mathrm{Arg}(-1-i) = -\frac{3\pi}{4}$

$$\therefore \quad -1-i = |z| e^{-i\frac{3\pi}{4}}$$

$$= \sqrt{2}\, e^{-i\frac{3\pi}{4}}$$

$$= \sqrt{2}\, \exp\left[-i\frac{3\pi}{4}\right] \quad \left(\begin{array}{c}\text{Another}\\ \text{notation}\\ \text{for } e^{i\theta}\end{array}\right)$$

Of course $-1-i = \sqrt{2}\, e^{i\frac{5\pi}{4}}$

$$= \sqrt{2}\, e^{i(-\frac{3\pi}{4} + 2k\pi)} \quad \forall k \in \mathbb{Z}.$$

<u>Notation</u>: $\boxed{z = |z| e^{i \arg z}}$

<u>Note</u>: We can represent a circle of radius R centered at z_0 by

$$z = z_0 + Re^{i\theta} \qquad \theta \in (-\pi, \pi]$$

§7 Products & Powers in Exponential form.

<u>Fact</u>: $$\boxed{e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}}$$

(Pf: Ex! by compound angle formula!)

Then, if $z_1 = r_1 e^{i\theta_1}$ & $z_2 = r_2 e^{i\theta_2}$,

- $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

  $\Rightarrow \quad |z_1 z_2| = |z_1||z_2|$

- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$ (provided $z_2 \neq 0$)

  in particular $z^{-1} = \frac{1}{z} = \frac{1}{r} e^{-i\theta}$

- $z^n = r^n e^{in\theta}$, $\forall n \in \mathbb{Z}$.

<u>de Moivre's formula</u>

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

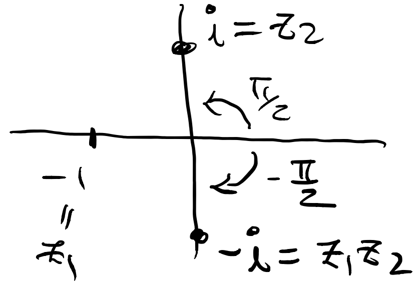
<u>Application</u>: $n=2 \Rightarrow \begin{cases} \cos 2\theta = \cos^2\theta - \sin^2\theta \\ \sin 2\theta = 2\sin\theta\cos\theta \end{cases}$

$n=3$ $\begin{cases} \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \\ \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta \end{cases}$ (Ex!)

§8 Arguments of Products and Quotients

eg: Let $z_1 = -1$, $z_2 = i$

then $\operatorname{Arg} z_1 = \pi$, $\operatorname{Arg} z_2 = \frac{\pi}{2}$ (Principal Argument)



& $z_1 z_2 = -i$

$\operatorname{Arg}(z_1 z_2) = -\frac{\pi}{2}$ (Principal Argument)

Consider $z_1 z_2 = e^{i\operatorname{Arg} z_1} e^{i\operatorname{Arg} z_2} = e^{i(\operatorname{Arg} z_1 + \operatorname{Arg} z_2)}$

$= e^{i\frac{3\pi}{2}}$

Note: $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 = \frac{3\pi}{2} \in \arg(z_1 z_2)$

but $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 \neq \operatorname{Arg}(z_1 z_2) = -\frac{\pi}{2}$

$\therefore$ Principal Argument + Principal Argument may not equal to the Principal Argument of the product.

We only have the following set equality:

$$\boxed{\arg z_1 + \arg z_2 = \arg(z_1 z_2)} \quad (\text{for } z_1, z_2 \neq 0)$$

(For sets S_1, S_2 in $\mathbb{R}$,

$$S_1 + S_2 \overset{\text{def}}{=\!=} \{a+b : a \in S_1, b \in S_2\}$$)

Similarly $$\boxed{\arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2} \quad (z_1, z_2 \neq 0)$$