Ch1 Complex Numbers

Standard notations:

$$\begin{cases} \mathbb{N} = \{0, 1, 2, 3, \cdots\} & \text{set of natural numbers} \\ \mathbb{Z} = \{\cdots -3, -2, -1, 0, 1, 2, 3, \cdots\} & \text{set of integers} \\ \mathbb{Q} = \text{set of rational numbers} \\ \mathbb{R} = \text{set of real numbers} \end{cases}$$

§1 Sums & Product

Def: The set of complex numbers $\mathbb{C}$ is

$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ (as set) with operations

defined by

$$(x_1, y_1) + (x_2, y_2) \overset{\text{def}}{=} (x_1 + x_2, y_1 + y_2)$$

$$(x_1, y_1)(x_2, y_2) \overset{\text{def}}{=} (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2)$$

[where $x_1 + x_2, \cdots, x_1 x_2, \cdots$ etc are addition & multiplication in $\mathbb{R}$.]

Notes: (1) If $z = (x, y)$, then $x = \operatorname{Re} z$ (real part)

$y = \operatorname{Im} z$ (imaginary part)

(2) "$+$" is called addition, "$\cdot$" multiplication.

(3) When restricted to $\mathbb{R}\times\{0\}=\{(x,0): x\in\mathbb{R}\}$

then $\begin{cases}(x_1,0)+(x_2,0)=(x_1+x_2,0)\\(x_1,0)(x_2,0)=(x_1x_2,0)\end{cases}$

Same operations as in $\mathbb{R}$.

So $(\mathbb{C},+,\cdot)$ is a natural extension of $(\mathbb{R},+,\cdot)$ when $x\in(\mathbb{R},+,\cdot)$ interpreted as $(x,0)\in(\mathbb{C},+,\cdot)$

i.e. $\quad x \longleftrightarrow (x,0)$

(4) If we write x for $(x,0)$

i for $(0,1)$,

then
$$z=(x,y)=(x,0)+(0,y) \quad \text{(by dofn)}$$
$$=(x,0)+(0,1)(y,0) \quad \text{(check!)}$$
$$=x+iy$$

It is more convenient to use $\boxed{z=x+iy}$ to represent the cpx number $z=(x,y)$.

(5) In this notation:

$$\begin{cases}(x_1+iy_1)+(x_2+iy_2)=(x_1+x_2)+i(y_1+y_2)\\(x_1+iy_1)(x_2+iy_2)=(x_1x_2-y_1y_2)+i(y_1x_2+x_1y_2)\end{cases}$$

& $\quad i^2+1=0 \quad$ (Check!)

§2 Basic Algebraic Properties

$$\begin{cases} z_1+z_2 = z_2+z_1 \\ (z_1+z_2)+z_3 = z_1+(z_2+z_3) \\ \exists\ 0 \text{ s.t. } z+0=z,\ \forall z \\ \forall z,\ \exists\ -z \text{ s.t. } z+(-z)=0 \end{cases}$$

$$\begin{cases} z_1 z_2 = z_2 z_1 \\ (z_1 z_2) z_3 = z_1 (z_2 z_3) \\ \exists\ 1 \text{ s.t. } z1 = z \\ \forall z \neq 0,\ \exists\ z^{-1} \text{ s.t. } z z^{-1} = 1 \end{cases}$$

- $z(z_1+z_2) = zz_1 + zz_2$

<u>Notes</u>: (1) $\forall\, n \in \mathbb{Z}$, z^n is defined by induction:

$$\begin{cases} z^{n+1} = z^n z & \text{with} \\ z^0 = 1 \end{cases}$$

(2) $z_1 z_2 = 0 \Rightarrow z_1 = 0$ or $z_2 = 0$

(3) <u>Subtraction</u> $z_1 - z_2 \overset{\text{def}}{=} z_1 + (-z_2)$

<u>Division</u> $\dfrac{z_1}{z_2} \overset{\text{def}}{=} z_1 z_2^{-1}$ (for $z_2 \neq 0$)

(4) <u>Binomial formula</u> is also hold for cpx numbers.

$$(z_1+z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k}$$

where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ $(k=0,1,2,\cdots)$

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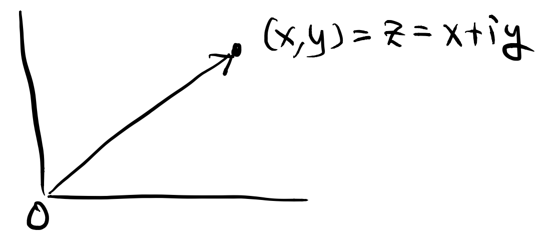
§4 Vectors & Moduli

By definition of cpx numbers, $z = x+iy$ is naturally identified as a plane vector (x,y) in $\mathbb{R}^2$



Notes: (1) cpx number addition coincides with vector addition

$$z_1 + z_2 = (x_1+x_2) + i(y_1+y_2)$$

$$\updownarrow$$

$$(x_1,y_1)+(x_2,y_2) = (x_1+x_2,\ y_1+y_2)$$

$\uparrow$ vector addition

(2) But cpx number multiplication is neither the scalar product nor vector product (in vector analysis)

(i) scalar product $\alpha(x,y) = (\alpha x, \alpha y)$ defined only for $\alpha \in \mathbb{R}$

- cpx multiplication is an extension to allow $\alpha \in \mathbb{C}$.

(ii) vector product (in vector analysis) takes 2 plane vectors to a vector out of the plane.

Def: The modulus, or absolute value, of $z = x+iy$ is defined by

$$|z| = \sqrt{x^2+y^2}$$

(non-negative real root of the real number x^2+y^2)

i.e. $|z|$ = length of the vector (x,y)

= distance between (x,y) & $(0,0)$

Notes: (1) The inequality $z_1 < z_2$ is not defined for cpx numbers as there is NO "inequality" on $\mathbb{C}$ that extends the "inequality on $\mathbb{R}$" and compatible with the algebraic operations.

Therefore $z_1 < z_2$ is meaningless unless $z_1, z_2 \in \mathbb{R}$,

However $|z_1| < |z_2|$ is meaningful!

(2) Easy to prove that

$$\begin{cases} \operatorname{Re} z \le |\operatorname{Re} z| \le |z| \\ \operatorname{Im} z \le |\operatorname{Im} z| \le |z| \end{cases}$$

(3) Triangle inequality

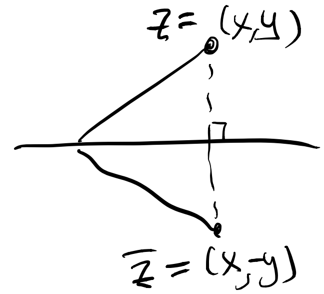
$$|z_1+z_2| \leq |z_1|+|z_2|$$

& hence $\big||z_1|-|z_2|\big| \leq |z_1 \pm z_2| \leq |z_1|+|z_2|$ (Ex!)

§5 Complex Conjugate

Def: The complex conjugate (or simply conjugate) of $z = x+iy$ is

$$\boxed{\bar{z} = x - iy}$$



i.e. $\bar{z}$ is represented by the reflection in real axis.

It is easy to prove (Ex!)

- $\begin{cases} \overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2 \\ \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2 , \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \end{cases}$

- $\begin{cases} \mathrm{Re}\, z = \dfrac{z+\bar{z}}{2} \\ \mathrm{Im}\, z = \dfrac{z-\bar{z}}{2i} \end{cases}$

- $z\bar{z} = |z|^2$

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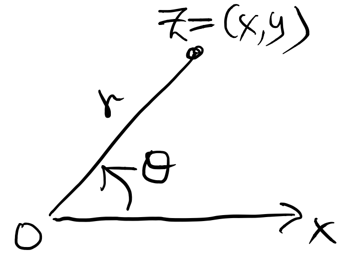
§6 Exponential Form

Using polar coordinate (r,θ) for (x,y) for each $z \neq 0$, we can write



$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$, & for some $\theta \in \mathbb{R}$.

Notes (1) θ is undefined for $z=0$

(2) θ is only defined up $2k\pi$, $k \in \mathbb{Z}$.

i.e. $\theta + 2k\pi$, for $k \in \mathbb{Z}$, also satisfies

$$z = r\left[\cos(\theta+2k\pi) + i\sin(\theta+2k\pi)\right]$$

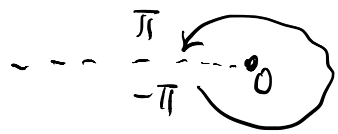
Definitions

(1) Each value of θ s.t. $z = |z|(\cos\theta + i\sin\theta)$ is called an argument of z.

(2) $\boxed{\arg z = \text{set of all arguments of } z}$

(3) The __principal value__ of $\arg z$, or __principal argument__ of z, denoted by $\underline{\operatorname{Arg} z}$ is the value

$\Theta \in \arg z$ s.t. $\underline{\underline{-\pi < \Theta \leq \pi}}$.



($\operatorname{Arg} z$ is discontinuous along negative real axis.)

i.e.

$$\begin{cases} \arg z = \{ \operatorname{Arg} z + 2k\pi : k \in \mathbb{Z} \} & \text{is a set.} \\ \quad\;\;\; = \operatorname{Arg} z + 2k\pi & \text{for simplicity} \\ \operatorname{Arg} z \in (-\pi, \pi] & \end{cases}$$