Pf of (ii) By the note in Step 1, we have

(since φ = local isom)

$$\begin{tikzcd} B^i(\delta) \arrow[r, "d\varphi"] \arrow[d, "\exp^M_{x_i}"'] \arrow[dr, phantom, "\circlearrowright"] & B^N(\delta) \arrow[d, "\exp^N_y"] \\ B^i_\delta \arrow[r, "\varphi"] & B^N_\delta \end{tikzcd}$$

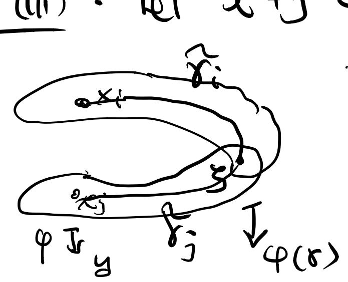
i.e. $\quad \varphi \circ \exp^M_{x_i} = \exp^N_y \circ d\varphi$

By the choice of $\delta > 0$, $\exp^N_y$ and $d\varphi$ are diffeomorphisms. Hence $\exp^M_{x_i}$ has to be an immersion. On the other hand $\exp^M_{x_i} : B^i(\delta) \to B^i_\delta$ is surjective (since M is complete), therefore we have

$$\varphi = \exp^N_y \circ d\varphi \circ (\exp^M_{x_i})^{-1}$$

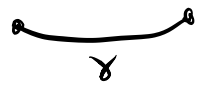
is a diffeomorphism. This proves (ii).

Pf of (iii) : let $i \neq j \in \Lambda$. Suppose that $B^i_\delta \cap B^j_\delta \neq \phi$



Then $\exists\ \xi \in B^i_\delta \cap B^j_\delta$.

Using (ii), $\exists$ geodesics



$\tilde{\gamma}_i \in B_\delta^i$ & $\tilde{\gamma}_j \in B_\delta^j$ $\begin{pmatrix} \tilde{\gamma}_i(0) = \xi = \tilde{\gamma}_j(0) \\ \tilde{\gamma}_i(1) = x_i \\ \tilde{\gamma}_j(1) = x_j \end{pmatrix}$

joining ξ to x_i and x_j respectively.

Then $\varphi(\tilde{\gamma}_i)$ & $\varphi(\tilde{\gamma}_j)$ are geodesics in B_δ^N

joining $\varphi(\xi)$ and $y = \varphi(x_i) = \varphi(x_j)$.

$\Rightarrow$ $\varphi(\tilde{\gamma}_i) = \varphi(\tilde{\gamma}_j) = \gamma$ the unique geodesic in B_δ^N

joining $\varphi(\xi)$ to y.

Therefore $\tilde{\gamma}_i, \tilde{\gamma}_j$ are liftings of γ passing through a common point ξ, we have $\tilde{\gamma}_i = \tilde{\gamma}_j$.

$\Rightarrow$ $x_i = \tilde{\gamma}_i(1) = \tilde{\gamma}_j(1) = x_j$. Contradiction

This proves (iii).

By this claim, B_δ^N is the required (uniform) nbd. of y. $\therefore$ φ is a covering map. ⁂

<u>Lemma 9</u>: Let
- M = complete Riem mfd.
- $x \in M$ s.t.
- $\exp_x : T_xM \to M$ has no conjugate point.

Then $\exp_x$ is a covering map.

Pf: Let g = Riem. metric on M

Denote $\tilde{g} = (\exp_x)^* g$ be the pull-back metric of g by $\exp_x$ on T_xM (since $\exp_x$ has no conjugate point), i.e.

$$\boxed{\tilde{g}(X, Y) \overset{def}{=} g((d\exp_x)(X), (d\exp_x)(Y))}$$

$\forall X, Y \in \Gamma(T_xM)$

Claim: $\tilde{g}$ is a complete metric on T_xM.

Pf of Claim: Note that Euclidean rays (from 0) in T_xM can be parametrized by

$$\begin{aligned} \tilde{\gamma} : [0, \infty) &\to T_xM \\ \rotatebox{90}{$\in$} \quad & \quad\ \rotatebox{90}{$\in$} \\ t &\longmapsto tv \end{aligned} \quad (\text{for some } v \in T_xM)$$

By definition of $\exp_x$, $\exp_x(\tilde{\gamma}(t))$ is a geodesic in M starting at x. Therefore, by definition of $\tilde{g} = (\exp_x)^* g$, $\tilde{\gamma}(t)$ is a geodesic of $(T_xM, \tilde{g})$ starting from 0. This implies geodesics from

$0 \in T_xM$ are defined $\forall t \in [0,\infty)$ (Since M is complete). Hence

$$\exp_0^{(T_xM,\tilde{g})} : T_0(T_xM) \to (T_xM, \tilde{g})$$

is defined on the whole $T_0(T_xM)$. Hopf-Rinow Thm $\Rightarrow$ $(T_xM, \tilde{g})$ is complete.

This proves the claim.

Now by the claim and the assumption that $\exp_x$ has no conjugate point, $\exp_x : (T_xM, \tilde{g}) \to (M, g)$ is a local isometry from a complete Riem. manifold. Therefore, lemma 8 $\Rightarrow$ $\exp_x : T_xM \to M$ is a covering. ※

Pf of (2) of Cartan-Hadamard

By lemma 9, $\exp_x : T_xM \to M$ is a covering. Together with the assumption that M is simply-connected, we have proved that $\exp_x$ is a diffeo. ※

Thm 10: Let M, N = simply-connected n-dim'l space forms with constant sectional curvature K. ($n \geq 2$) Let $x \in M$, $y \in N$ and $\{e_1, \cdots, e_n\} \subset T_xM$ and $\{\varepsilon_1, \cdots, \varepsilon_n\} \subset T_yN$ are orthonormal basis respectively.

Then $\exists$ unique isometry $\varphi: M \to N$ such that

$$\begin{cases} \varphi(x) = y \quad \text{and} \\ d\varphi(e_i) = \varepsilon_i, \quad \forall i \end{cases}$$

Note: Thm 10 $\Rightarrow$ uniqueness of the Thm 1 in Ch5.

We need the following Lemmas 11 & 12:

Lemma 11 Let
- M = n-dim'l space form
- K = constant sectional curvature
- $x \in M$, $\{e_1, \cdots, e_n\} \subset T_xM$ orthonormal basis.

Then the curvature tensor satisfies

$$\boxed{R_{e_i e_j} e_k = K(\delta_{ik} e_j - \delta_{jk} e_i), \quad \forall i, j, k = 1, \cdots, n.}$$

Pf: Define $\tilde{R}$ by the RHS, i.e.

$$\tilde{R}_{e_ie_j}e_k \stackrel{\text{def}}{=} K(\delta_{ik}e_j - \delta_{jk}e_i)$$

Then $\tilde{R}$ can be extended to a tensor (Ex!) satisfying all the symmetric properties of the curvature tensor (i.e. (1)-(4) in Lemma 1 of §3.3) (Ex!)

Furthermore, $\forall$ tangent vectors v & w with $|v|=|w|=1$ and $\langle v,w\rangle=0$, one has

$$\langle \tilde{R}_{vw}v, w\rangle = K \quad \text{(Ex!)}$$

Therefore Lemma 2 of §3.3 $\Rightarrow \tilde{R}=R$. #

Lemma 12: Same assumption as in Lemma 11

Let
- $v \in T_xM$ with $|v|=1$
- $v^\perp$ = orthogonal complement of v

Then $$R_{vw}v = \begin{cases} Kw, & \text{if } w \in v^\perp \\ 0, & \text{if } w = cv, \text{ for some } c \in \mathbb{R}. \end{cases}$$

(Pf: Immediately from Lemma 11.)

<u>Pf of Thm 10</u> : It is clear that we only need to show the cases of $K = 0, +1$ or -1. And we may assume $M = \mathbb{R}^n, S^n$ or $\mathbb{H}^n$.

<u>Case 1</u> : $K = 0$ or -1.

Since $K \leq 0$, Cartan-Hadamard $\Rightarrow$

$$\left.\begin{array}{l} \exp_x^M : T_xM \to M \\ \exp_y^N : T_yN \to N \end{array}\right\} \text{ are diffeomorphisms.}$$

Let $\Phi : T_xM \to T_yN$ be the unique isometry between the <u>inner product spaces</u> T_xM & T_yN such that

$$\begin{tikzcd} T_xM \arrow[r, "\Phi"] \arrow[d, "\exp_x^M"'] \arrow[dr, phantom, "\circlearrowright"] & T_yN \arrow[d, "\exp_y^N"] \\ M \arrow[r, dashed, "\varphi"'] & N \end{tikzcd}$$

$$\Phi(e_i) = \varepsilon_i, \ \forall i = 1, \cdots, n.$$

Define $\varphi : M \to N$ by $\varphi = \exp_y^N \circ \Phi \circ (\exp_x^M)^{-1}$.

Clearly φ is a diffeomophism. We only need to show that φ is an isometry. i.e.

$\forall\, z \in M$ and $X \in T_z M$ we have

$$|d\varphi(X)|_N = |X|_M$$

By Cartan-Hadamard,

$\exists\; T \in T_x M$ and $w \in T_T(T_x M) \cong T_x M$ s.t.

$$z = \exp_x^M(T) \quad \text{and} \quad X = (d\exp_x^M)_T(w)$$

Then we can define a 1-parameter family of geodesics

$$\gamma_u(t) = \exp_x^M[t(T + uw)]$$

Let $U(t)$ = transversal vector field of γ_u along γ_0. Then $U(t)$ is a Jacobi field s.t.

$$\begin{cases} U(0) = 0 \\ U'(0) = w \end{cases}$$

and further $U(1) = (d\exp_x^M)_T(w) = X$.

In N, we define correspondingly

$$\gamma_u^N(t) = \exp_y^N\left[t\left(\Phi(T) + u\,\Phi(w)\right)\right]$$

& $\upsilon^N(t)$ = transversal vector field of $\{\gamma_u^N\}$ along γ_0^N

Then υ^N is a Jacobi field along $\gamma_0^N \subset N$

s.t. $\begin{cases} \upsilon^N(0) = 0 \\ (\upsilon^N)'(0) = \Phi(w) \end{cases}$

Note that

$$\begin{aligned}
\varphi(\gamma_u(t)) &= \left[\exp_y^N \circ \Phi \circ (\exp_x^M)^{-1}\right]\left(\exp_x^M[t(T+uw)]\right) \\
&= \exp_y^N \circ \Phi\,(t(T+uw)) \\
&= \exp_y^N\left[t\left(\Phi(T) + u\,\Phi(w)\right)\right] \\
&= \gamma_u^N(t)
\end{aligned}$$

$\Rightarrow \quad d\varphi(\upsilon(t)) = \upsilon^N(t) \qquad$ (by differentiation)

$\Rightarrow \quad \upsilon^N(1) = d\varphi(\upsilon(1)) = d\varphi(X)$.

Therefore, we need to show that

$$|v^N(1)|_N = |v(1)|_M .$$

To see this, we use parallel orthonormal frames $\{e_1(t), \cdots, e_n(t)\}$ & $\{\varepsilon_1(t), \cdots, \varepsilon_n(t)\}$ along γ_0 and γ_0^N respectively such that

$$\begin{cases} e_i(0) = e_i \\ \varepsilon_i(0) = \varepsilon_i \end{cases} \quad \forall\, i = 1, \cdots, n .$$

Then $\begin{cases} v(t) = \sum_i f_i(t)\, e_i(t) \\ v^N(t) = \sum_i g_i(t)\, \varepsilon_i(t) \end{cases}$ for some function $f_i(t), g_i(t)$.

Furthermore, $v(0) = 0$ & $v'(0) = w \Rightarrow$

$$\begin{cases} f_i(0) = 0 \\ f_i'(0) = \langle w, e_i \rangle \end{cases}$$

(by lemma K) (later)

$$\therefore \quad (*) \begin{cases} f_i'' + \sum_i f_i \cdot K\left[|T|^2 \delta_{ij} - \langle T, e_i \rangle \langle T, e_j \rangle \right] = 0 \\ f_i(0) = 0 \\ f_i'(0) = \langle w, e_i \rangle \end{cases}$$

Similarly, we have

$$\begin{cases} g_i'' + \sum_i g_i \cdot K\left[|\Phi(T)|^2\delta_{ij} - \langle \Phi(T), \varepsilon_i\rangle\langle \Phi(T), \varepsilon_j\rangle\right] = 0 \\ g_i(0) = 0 \\ g_i'(0) = \langle \Phi(w), \varepsilon_i\rangle \end{cases}$$

Using the fact that Φ is an isometry (between inner product spas T_xM & T_yN), we have

$$\begin{cases} |\Phi(T)|^2 = |T|^2 \\ \langle \Phi(T), \varepsilon_i\rangle = \langle \Phi(T), \Phi(e_i)\rangle = \langle T, e_i\rangle \\ \langle \Phi(w), \varepsilon_i\rangle = \langle w, e_i\rangle \end{cases}$$

$\therefore$ $\{f_i\}$ & $\{g_i\}$ satisfy the same IVP of an ODE system $(*)$, therefore $f_i \equiv g_i$, $\forall\, i = 1, \cdots, n$

Hence $|V^N(1)|^2 = \sum_i g_i^2(1) = \sum_i f_i^2(1) = |V(1)|^2$,

This proves the case that $K = 0$ or -1.

<u>Pf of $(*)$</u>:

We need to calculate the curvature term

$$R_{\gamma_0'(t)\,\upsilon(t)}\,\gamma_0'(t)$$

Let $V_0(t) = \dfrac{\gamma_0'(t)}{|\gamma_0'(t)|}$, then

$$R_{\gamma_0'(t)\,\upsilon(t)}\,\gamma_0'(t) = |\gamma_0'(t)|^2 R_{V_0(t)\,\upsilon(t)}\,V_0(t)$$

$$(\text{lemma 12}) = |\gamma_0'(t)|^2 K\left[\upsilon(t) - \langle \upsilon(t), V_0(t)\rangle V_0(t)\right]$$

Since $\langle \gamma_0'(t), \gamma_0'(t)\rangle = \langle \gamma_0'(0), \gamma_0'(0)\rangle = |T|^2$

$$\langle \gamma_0'(t), e_i(t)\rangle = \langle T, e_i\rangle,$$

we have
$$\upsilon''(t) + R_{\gamma_0'(t)\,\upsilon(t)}\,\gamma_0'(t) = 0$$

$$\Leftrightarrow \sum f_i'' e_i + |\gamma_0'|^2 K\left[\sum f_i e_i - \frac{\langle \sum f_i e_i, \gamma_0'\rangle}{|\gamma_0'|^2}\gamma_0'\right] = 0$$

$$\Leftrightarrow \sum_i \left(f_i'' e_i + |T|^2 K f_i\right) e_i - K \sum_i f_i \langle e_i, \gamma_0'\rangle \gamma_0' = 0$$

$$\Leftrightarrow \sum_i \left(f_i'' e_i + |T|^2 K f_i\right) e_i - K \sum_i f_i \langle e_i, T\rangle \sum_j \langle e_j, \gamma_0'\rangle e_j = 0$$

$$\Leftrightarrow \sum_i \left(f_i'' e_i + |T|^2 K f_i\right) e_i - K \sum_{i,j} f_{\underset{j}{\cancel{i}}} \langle e_{\underset{j}{\cancel{i}}}, T\rangle \langle e_{\underset{i}{\cancel{j}}}, T\rangle\, e_{\underset{i}{\cancel{j}}} = 0$$

$$\Leftrightarrow \sum_i \Big[f_i'' e_i + |T|^2 K f_i - K \sum_j f_j \langle e_j, T\rangle \langle e_i, T\rangle \Big] e_i = 0$$

$$\Leftrightarrow \quad f_i'' + \sum_j f_j K \left[|T|^2 \delta_{ij} - \langle e_j, T\rangle \langle e_i, T\rangle \right] = 0$$

$\forall\, i = 1, \cdots, n.$

#

<u>Case of $K = +1$</u>

We may assume $M = S^n$.

If $\bar{x} = -x$ (the antipodal point of x), then

$(\exp_x^M)^{-1} : S^n \setminus \{\bar{x}\} \to T_x S^n$ is well-defined.

Therefore, we can define similarly the map

$$\varphi = \exp_y^N \circ \Phi \circ (\exp_x^M)^{-1} : S^n \setminus \{\bar{x}\} \to N$$

Similar argument shows that φ is a <u>local</u> isometry.

Observes that $\forall\, z \in S^n \setminus \{x, \bar{x}\}$, we still have

$$\begin{tikzcd} T_{\bar z}S^n \arrow[r, "d\varphi"] \arrow[dr, phantom, "\circlearrowright"] & T_{\varphi(\bar z)}N \arrow[d, "\exp^N_{\varphi(\bar z)}"] \\ S^n\setminus\{\bar x, \bar z\} \arrow[u, "(\exp^{S^n}_{\bar z})^{-1}"] \arrow[r, "\varphi"] & N \end{tikzcd}$$

(since φ is a local isom.)

Note that $\Psi = d\varphi|_{T_{\bar z}S^n} : T_{\bar z}S^n \to T_{\varphi(\bar z)}N$ is an inner product space isometry, same argument above implies that $\psi : S^n\setminus\{\bar z\} \to N$ defined by

$$\psi \overset{\text{def}}{=} \exp^N_{\varphi(\bar z)} \circ \Psi \circ (\exp^{S^n}_{\bar z})^{-1}$$

is a local isometry. By the above commutative diagram, $\forall\, p \in S^n\setminus\{\bar x, \bar z\}$,

$$\begin{aligned}\varphi(p) &= \exp^N_{\varphi(\bar z)} \circ d\varphi \circ (\exp^{S^n}_{\bar z})^{-1}(p) \\ &= \exp^N_{\varphi(\bar z)} \circ \left(d\varphi|_{T_{\bar z}S^n}\right) \circ (\exp^{S^n}_{\bar z})^{-1}(p) \\ &= \exp^N_{\varphi(\bar z)} \circ \Psi \circ (\exp^{S^n}_{\bar z})^{-1}(p) = \psi(p)\end{aligned}$$

$\Rightarrow$ we can extend φ to be defined on the whole S^n by setting $\varphi(\bar{x}) = \psi(\bar{x})$.

Then by the construction of $\varphi: S^n \to N$ is a local isometry. Similar argument as in Lemma 8 $\Rightarrow$ φ is a covering map

$\Rightarrow$ φ is an isometry, since N is simply-connected.

Finally, it is clear that $d\varphi(e_i) = \varepsilon_i$, $\forall i = 1, \cdots, n$

So we've proved the existence part of Thm 10.

For uniqueness, we prove the following

<u>Lemma 13</u>: Let $\varphi_i : M \to N$, $i = 1, 2$, be 2 local isometries between complete Riem. mfds M & N (connected) such that for some $x \in M$,

$$\begin{cases} \varphi_1(x) = \varphi_2(x) \\ d\varphi_1|_{T_xM} = d\varphi_2|_{T_xM} \end{cases}.$$

Then $\varphi_1 \equiv \varphi_2$.

Pf of Uniqueness of Thm 10 : Immediately from Lemma 13.

#

Pf of Lemma 13 :

Let $S = \{ z \in M : \varphi_1(z) = \varphi_2(z) \ \& \ d\varphi_1|_{T_zM} = d\varphi_2|_{T_zM} \}$.

- By assumption, $x \in S$. $\therefore S \neq \phi$.
- It is clear that S is closed by continuity.
- If $z \in S$, take $\delta > 0$ s.t.

  $\exp_z^M : B(\delta) \to M$ is a diffeo. injection.

Recall that we have

$$\begin{array}{ccc} T_zM & \xrightarrow{d\varphi} & T_{\varphi(z)}N \\ \exp_z^M \downarrow & \circlearrowleft & \downarrow \exp_{\varphi(z)}^N \\ M & \xrightarrow{\varphi} & N \end{array}$$

$\forall$ local isometry φ.

Applying this to φ_1 & φ_2, we have

$$\exp_z^M(B(\delta)) \subset S \qquad (Ex!)$$

$\Rightarrow S$ is open.

Therefore, by connectedness of $M \Rightarrow S = M$. ※

Cor 14: Let M = complete simply-connected Riem. mfd. of dim n.

Then M is a space form

$\Leftrightarrow \forall x, y \in M$ and

$\forall$ orthonormal basis $\{e_i\}$ of T_xM &

" " $\{\varepsilon_i\}$ of T_yM,

$\exists$ isometry $\varphi: M \to M$ s.t.

$\varphi(x) = y$ & $d\varphi(e_i) = \varepsilon_i$, $\forall i = 1, \cdots, n$.

(Pf: Immediately from Thm 10)

Note: Cor 14 proves that simply-connected space form is homogeneous.

In fact, we have more

Cor 15: Simply-connected space forms are two-point homogeneous.

__Def__: M is called __two-points homogeneous__ if

$\forall\, p_1, p_2, q_1, q_2 \in M$ with

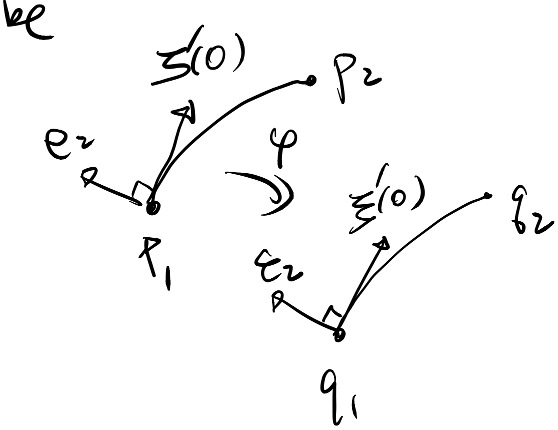
$$d(p_1,p_2) = d(q_1,q_2)$$

$\exists$ an isometry $\varphi : M \to M$ such that

$$\varphi(p_1) = q_1 \ \& \ \varphi(p_2) = q_2 .$$

__Pf__: Let p_1, p_2, q_1, q_2 as in the Thm.

Let $\zeta, \xi : [0,\alpha] \to M$ be normalized geodesics s.t.



$\zeta(0) = p_1,\ \zeta(\alpha) = p_2$

$\xi(0) = q_1,\ \xi(\alpha) = q_2$

$(\alpha = d(p_1,p_2) = d(q_1,q_2))$

Choose orthonormal bases

$\{e_i\}$ on $T_{p_1}M$ s.t. $e_1 = \zeta'(0)$

$\{\varepsilon_i\}$ on $T_{q_1}M$ s.t. $\varepsilon_1 = \xi'(0)$

Then Thm 10 (a Cor 14) $\Rightarrow$ $\exists$ isometry

$\varphi: M \to M$ s.t.

$$\varphi(p_1) = q_1, \quad d\varphi(e_i) = \varepsilon_i$$

$\Rightarrow$ $\varphi \circ \zeta$ & ξ are geodesics with the same initial data, hence $\varphi \circ \zeta = \xi$.

$\Rightarrow$ $\varphi(p_2) = q_2$ ※