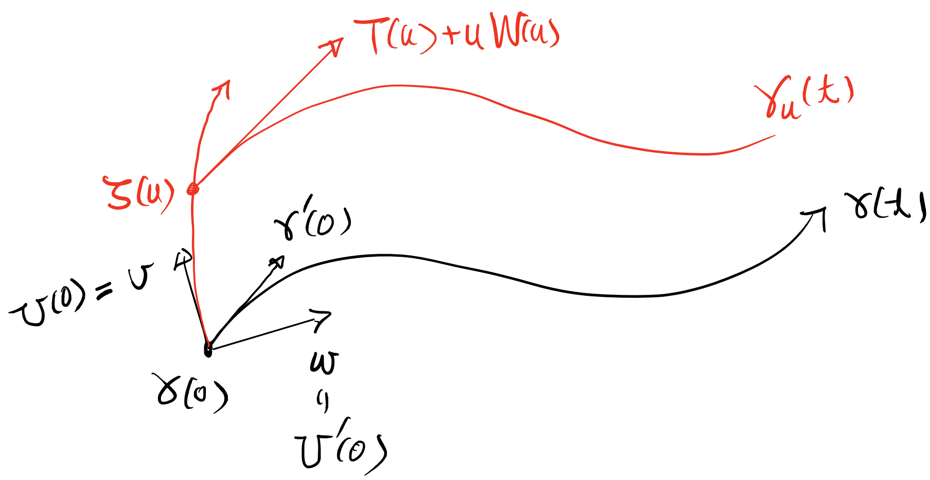
Pf of lemma 2 ($\Rightarrow$)

Let U be a Jacobi field along γ with

$$\begin{cases} U(0) = v \\ U'(0) = w \end{cases} \qquad \left(\text{by identifying } T_{\vec{p}}(T_{\gamma(0)}M) \cong T_{\gamma(0)}M\right)$$



let $\zeta : [0,\underset{\substack{\Downarrow \\ u}}{\varepsilon}] \to M$ be a geodesic such that

$$\zeta(0) = \gamma(0) \quad \text{and} \quad \gamma'(0) = v$$

Define parallel vector fields $T(u)$ and $W(u)$ for $u \in [0,\varepsilon]$ along ζ such that $T(0) = \gamma'(0)$ and $W(0) = w$.

$$\Gamma(t,u) = \gamma_u(t) = \exp_{\zeta(u)}\left[t\left(T(u) + uW(u)\right)\right], \quad \forall u \in [0,\varepsilon].$$

Let U_1 = transversal vector field of γ_u along $\gamma = \gamma_0$

Then U_1 is a Jacobi field.

And $U_1(0) = \frac{\partial}{\partial u}\Big|_{u=0} \gamma_u(0)$

$$= \frac{\partial}{\partial u}\Big|_{u=0} \exp_{\zeta(u)}(0)$$

$$= \frac{d}{du}\Big|_{u=0} \zeta(u)$$

$$= \zeta'(0) = v.$$

Since $T_1 = d\Gamma(\frac{\partial}{\partial t})$ is a vector field along Γ and when restricted to γ, we have

$$[T_1, U_1] = 0$$

Hence $U_1'(0) = D_{\gamma'(0)} U_1 = D_{U_1(0)} T_1$ (since $[T_1, U_1] = 0$)

$$= D_v T_1 = D_{\zeta'(0)} T_1$$

Note that

$$T_1(\zeta(u)) = \frac{\partial}{\partial t}\Big|_{t=0} \exp_{\zeta(u)}[t(T(u) + uW(u))]$$

$$= T(u) + uW(u)$$

$\Rightarrow$ $U_1'(0) = D_{\zeta'(0)} T_1 = D_{\zeta'(0)}[T(u) + uW(u)]$

$$= W(0) = w \text{ (as } T, W \text{ parallel along } \zeta)$$

Altogether, $U(0)=U_1(0)$ & $U'(0)=U_1'(0)$

Uniqueness of Jacobi field (with initial data)

$$\Rightarrow \quad U = U_1 = \text{transversal vector field.}$$

$\#$

<u>Lemma 3</u>: Let U be a Jacobi field along a geodesic γ.

Then $\exists$ constants a, b such that

$$U = U^{\perp} + (at+b)\gamma'$$

where $U^{\perp}$ is a Jacobi field s.t. $\langle U^{\perp},\gamma'\rangle = 0$, $\forall t$.

<u>Pf</u>: Consider

$$\begin{aligned}
\frac{d^2}{dt^2}\langle U,\gamma'\rangle &= \frac{d}{dt}\left(D_{\gamma'}\langle U,\gamma'\rangle\right) \\
&= \frac{d}{dt}\left(\langle D_{\gamma'}U,\gamma'\rangle + \langle U, \cancelto{0}{D_{\gamma'}\gamma'}\rangle\right) \\
&= \langle D_{\gamma'}D_{\gamma'}U,\gamma'\rangle \\
&= -\langle R_{\gamma' U}\gamma',\gamma'\rangle = 0
\end{aligned}$$

$\Rightarrow \quad \langle U,\gamma'\rangle = \tilde{a}t + \tilde{b}$ for some const. $\tilde{a}$ & $\tilde{b}$

Let
$$\begin{aligned}
U^{\perp} &= U - \left\langle U, \frac{\gamma'}{|\gamma'|}\right\rangle \frac{\gamma'}{|\gamma'|} \\
&= U - \left(\frac{\tilde{a}}{|\gamma'|^2}t + \frac{\tilde{b}}{|\gamma'|^2}\right)\gamma'
\end{aligned}$$

Since $|\gamma'| \equiv$ const., $\quad U^\perp = U - (at+b)\gamma'$

where $\quad a = \dfrac{\tilde{a}}{|\gamma'|^2}, \quad b = \dfrac{\tilde{b}}{|\gamma'|^2}$ are constants,

and $\quad \langle U^\perp, \gamma' \rangle = 0.$

Finally, $\quad (U^\perp)'' = U'' - [(at+b)\gamma']''$

$$= U'' = -R_{\gamma' U}\gamma'$$

$$= -R_{\gamma' U^\perp}\gamma' - (at+b)\cancel{R_{\gamma'\gamma'}}^{\,0}\gamma'$$

$\therefore \quad U^\perp$ is a Jacobi field. #

<u>Lemma 4</u>: If U is a Jacobi field along a geodesic γ such that $\quad \langle U(t_1), \gamma'(t_1) \rangle = \langle U(t_2), \gamma'(t_2) \rangle = 0$ for 2 different t_1 & t_2. Then $\langle U(t), \gamma'(t) \rangle = 0, \forall t.$

(Pf: Since $\langle U(t), \gamma'(t) \rangle$ is linear in t.) #

In summary, we have the following facts of Jacobi field

(A) Let $\zeta : [0, \varepsilon] \to M$ be a geodesic (curve) in M
$u \mapsto \zeta(u)$

$T(u)$, $W(u)$ parallel vector field along ζ.

Then $\boxed{\gamma_u(t) = \exp_{\zeta(u)} [t\, T(u) + u W(u)]}$

determines a 1-para. family of geodesic $\{\gamma_u\}$ s.t. its transversal vector field $U(t)$ along γ_0 is a Jacobi field with $\begin{cases} U(0) = \zeta'(0) \\ U'(0) = W(0). \end{cases}$

(B) [If we take $\zeta(u) = x \in M$ (const. curve) in (A)]

$\forall\, x \in M$; $T, w \in T_x M$. Then the 1-para. family of geodesics $\{\gamma_u\}$ defined by

$$\boxed{\gamma_u(t) = \exp_x [t(T + u w)]}$$

has a transversal vector field $U(t)$ s.t. $U(t)$ is a Jacobi field with $\begin{cases} U(0) = 0 \\ U'(0) = w \end{cases}$

(C) [Furthermore, adding condition $\langle T, w\rangle = 0$ to (B)]

Let $x \in M$; $T, w \in T_xM$ s.t. $\langle T, w\rangle = 0$

Let

$$\boxed{\gamma_u(t) = \exp_x[t(T+uw)]}$$

Then the transversal vector field $V(t)$ of $\{\gamma_u\}$ is a normal Jacobi field with $\begin{cases} V(0) = 0 \\ V'(0) = w \end{cases}$

(normal Jacobi field = Jacobi field normal to the geodesic)

Pf of (C)

We need

Lemma 5 (Gauss Lemma)

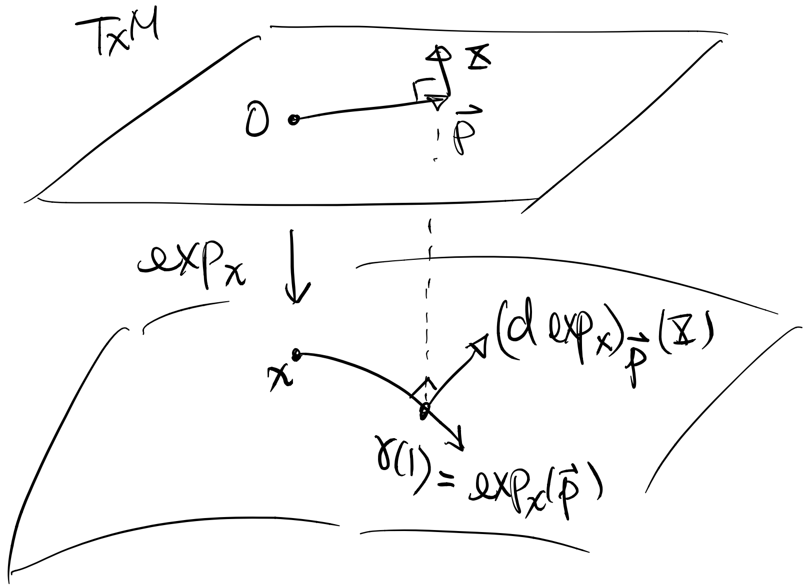
Let M be complete, $x \in M$, $\vec{p} \in T_xM$,

$$X \in T_{\vec{p}}(T_xM) \cong T_xM.$$

If $\langle \vec{p}, X\rangle = 0$, then

$$\langle (d\exp_x)_{\vec{p}}(X), \gamma'(1)\rangle = 0$$

where $\gamma : [0,1] \longrightarrow M$, $t \longmapsto \exp_x(t\vec{p})$



Pf: let $\xi : [0,\varepsilon] \to T_xM$ be a curve in T_xM s.t.

$$\xi(0) = \vec{p}, \quad \xi'(0) = X,$$

and that $\xi([0,\varepsilon]) \subset S^{n-1}_{|\vec{p}|} \subset T_xM$.

Such ξ exists since $X \perp \vec{p} \Rightarrow X \in T_{\vec{p}} S^{n-1}_{|\vec{p}|}$.

Consider
$$\begin{aligned} \Gamma : [0,1] \times [0,\varepsilon] &\to M \\ (t,u) &\mapsto \exp_x[t\xi(u)] \end{aligned}$$

Let $T = d\Gamma(\frac{\partial}{\partial t})$, $U = d\Gamma(\frac{\partial}{\partial u})$

Then $\gamma(t) = \Gamma(t,0)$, $\gamma'(1) = T(\gamma(1))$

$$(d\exp_x)_{\vec{p}}(X) = U(\gamma(1))$$

Since $|\xi(u)| = |\vec{p}|,\ \forall u \in [0,\varepsilon]$,

we have $\langle T, T\rangle = |\vec{p}|^2$ (geodesic has const. speed)

$$\therefore\ T\langle U, T\rangle = \langle D_T U, T\rangle + \langle U, \cancel{D_T T}^{\,0}\rangle$$

$$= \langle D_U T, T\rangle \qquad ([U,T]=0)$$

$$= \frac{1}{2} U\langle T, T\rangle = 0$$

$$\Rightarrow \langle U, T\rangle = \text{const. along } \gamma$$

$$= \lim_{t\to 0}\langle U(t), T(t)\rangle = \langle \cancel{U(0)}^{\,0}, T(0)\rangle$$

$$= 0. \ \#$$

<u>Pf of (C)</u>: Let $\xi : [0,\varepsilon] \to T_x M$

$$u \mapsto t(T+uw)$$

Then $\langle \xi'(0), \xi(0)\rangle = \langle tw, tT\rangle = t^2\langle w, T\rangle \overset{\text{by assumption}}{=} 0$

and $(d\exp_x)_{tT}(\xi'(0)) = U(t)$ $\left(\begin{array}{l} = \text{transversal} \\ \text{vector field of} \\ \exp_x[t(T+uw)] \end{array}\right)$

Consider the curve $\gamma : [0,1] \to M$

$$\tau \longmapsto \exp_x(\tau(tT))$$

Note that $\gamma_0(t) = \exp_x(tT)$ of the family $\exp_x[t(T+uw)]$

$$\Rightarrow \quad \gamma'(1) = \frac{d}{d\tau}\Big|_{\tau=1}\left[\exp_x(\tau(tT))\right] = (d\exp_x)_{(tT)}(tT)$$

$$= t\,(d\exp_x)_{(tT)}(T)$$

$$= t\,\gamma_0'(t) \qquad (\text{" ' " means derivative wrt } t)$$

Applying the Gauss Lemma to $\gamma(\tau)$ and $\vec{X} = \zeta'(0) = tw$

$$\vec{p} = \zeta(0) = tT, \quad \langle \vec{X}, \vec{p} \rangle = \langle \zeta'(0), \zeta(0) \rangle = 0.$$

we have

$$\langle v(t), \gamma_0'(t) \rangle = \langle (d\exp_x)_{(tT)}(\zeta'(0)), \frac{1}{t}\gamma'(1) \rangle$$

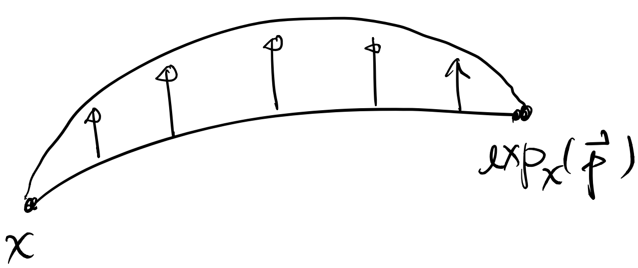
$$= \frac{1}{t}\langle (d\exp_x)_{(tT)}(\zeta'(0)), \gamma'(1) \rangle = 0$$

$\Rightarrow$ v is normal. ※

§6.2 Cartan-Hadamard Theorem

Lemma 6 $(d\exp_x)_{\vec{p}}$ is singular

$\iff \exists$ normal Jacobi field $V(t)$ along $\gamma(t) = \exp_x(t\vec{p})$, not identically zero, such that $V(0) = V(1) = 0$.



Pf: By the Lemma right before the original version of Gauss Lemma in Ch 4, $(d\exp_x)_{\vec{p}}$ is non-degenerate in the direction of $\vec{p}$. Therefore, we only need to consider X s.t. $\langle X, \vec{p} \rangle = 0$.

Let $X \in T_xM \cong T_{\vec{p}}(T_xM)$ s.t. $\langle X, \vec{p} \rangle = 0$

Then $\gamma_u(t) = \exp_x[t(\vec{p} + uX)]$

gives a normal Jacobi field with $V(0) = 0$

and $\mathcal{V}'(0) = \mathbb{X}$ (by fact (C)).

Furthermore $\mathcal{V}(1) = (d\exp_x)_{\vec{p}}(\mathbb{X})$

Therefore, if $\mathbb{X} \in \operatorname{Ker}(d\exp_x)_{\vec{p}}$ & $\mathbb{X} \neq 0$

then $\mathcal{V}(t)$ is a non-identically zero normal Jacobi field with $\mathcal{V}(0) = \mathcal{V}(1) = 0$.

This proves direction "$\Rightarrow$".

Conversely, any (non-identically zero) normal Jacobi field is the transversal vector field of a 1-para. family of geodesics given

$$\gamma_u(t) = \exp_{\zeta(u)}[t(T(u) + uW(u))]$$

with $\zeta(0) = \gamma(0)$, $\zeta'(0) = \mathcal{V}(0)$ &

T, W = parallel vector fields along $\zeta(u)$,

$\langle T, W \rangle = 0$.

Since $\mathcal{V}(0) = 0$, we may take $\zeta(u) \equiv \gamma(0) = x$,

$T = \vec{p}$ and $W = \mathbb{X} = \mathcal{V}'(0) \neq 0$,

& $\langle T, W \rangle = 0$.

Therefore, $0 \overset{\text{assumption}}{=} V(1) = (d\exp_x)_{\vec{p}}(X)$

$\Rightarrow \quad \underset{\substack{\neq \\ 0}}{X} \in \operatorname{Ker}(d\exp_x)_{\vec{p}}$

$\Rightarrow \quad (d\exp_x)_{\vec{p}}$ is singular. ※

Def: If $(d\exp_x)_{\vec{p}}$ is singular, then $\vec{p}$ is called a conjugate point of the map $\exp_x$, and $\exp_x(\vec{p})$ is called a conjugate point of x along the geodesic $\gamma(t) = \exp_x(t\vec{p})$.

Thm 7 (Cartan-Hadamard)

(1) Let M be a complete Riemannian mfd. with nonpositive sectional curvature. Then $\forall x \in M$, $\exp_x : T_xM \to M$ has no conjugate point.

(2) If M is a simply-connected complete Riem. mfd. such that for some $x \in M$, $\exp_x : T_xM \to M$ has

no conjugate point, then $\exp_x : T_xM \to M$ is a diffeomorphism.

<u>Pf of (1)</u>: let V be a normal Jacobi field with $V(0)=0$ along a geodesic $\gamma : [0,\infty) \to M$ (since M complete).

Let $f(t) = \langle V(t), V(t) \rangle$ along γ, then

$$f'(t) = 2\langle V'(t), V(t)\rangle$$

$$\Rightarrow \quad f''(t) = 2\langle V', V'\rangle + 2\langle V'', V\rangle$$

$$= 2|V'|^2 - 2\langle R_{\gamma' V}\gamma', V\rangle$$

Since $\langle R_{\gamma' V}\gamma', V\rangle = K(\text{span}\langle \gamma', V\rangle)\,|\gamma' \wedge V|^2$

$$= K|\gamma'|^2|V|^2 \leq 0$$

(since $\langle \gamma', V\rangle = 0$)

$$\Rightarrow \quad f''(t) \geq 0, \quad \forall\, t \in [0,\infty).$$

Now suppose $\gamma(t_0)$ is a conjugate point of x along some geodesic $\gamma : [0,\infty) \to M$. Then

Lemma 6 $\Rightarrow$ $\exists$ non-trivial normal Jacobi field $V(t)$ along γ s.t.

$$V(0) = V(t_0) = 0.$$

Applying the above, $|V(t)|^2$ is convex in t

$$\Rightarrow \quad 0 \le |V(t)|^2 \le \max\{|V(0)|^2, |V(t_0)|^2\} = 0, \quad \forall\, t \in [0, t_0]$$

$\Rightarrow$ $V \equiv 0$ on $[0, t_0]$. Contradiction. $\divideontimes$

The proof of (2) is much longer and we need the following lemmas (8 & 9):

<u>Lemma 8</u>: Let $\varphi: M \to N$ be a <u>local</u> isometry between (connected) Riemannian manifolds M and N. If M is <u>complete</u>, then N is complete and φ is a covering map.

<u>Pf</u>: <u>Step 1</u>: <u>φ is surjective & N complete</u>

- "φ = local isometry" $\Rightarrow$ $\varphi(M)$ open in N.

- Suppose $\gamma \subset N$ is a geodesic such that
$$\gamma \cap \varphi(M) \neq \emptyset.$$
Then $\exists\, x \in M$ such that $\varphi(x)$ is a point on γ.
Since φ is a local isometry, then near the point x, $\varphi^{-1} \circ \gamma$ defines a geodesic segment in a nbd. of x in M (passing thro. the point x)
The completeness of M implies $\varphi^{-1} \circ \gamma$ extends to a geodesic $\tilde{\gamma} \subset M$ defined on $(-\infty, \infty)$.
By assumption on φ, we have $\varphi \circ \tilde{\gamma} = (-\infty, \infty) \to \varphi(M) \subset N$
is a geodesic on N passing thro. $\varphi(x)$, and in a nbd. of $0 \in (-\infty, \infty)$, $\varphi \circ \tilde{\gamma} = \varphi(\varphi^{-1} \circ \gamma)$
$$= \gamma.$$
In particular, $(\varphi \circ \tilde{\gamma})'(0) = \gamma'(0)$
Therefore, uniqueness of geodesic $\Rightarrow$ $\varphi \circ \tilde{\gamma} = \gamma$
$\therefore$ $\gamma \subset \varphi(M)$
So we've proved that if a geodesic segment γ in N

intersects $\varphi(M)$, then $\gamma \subset \varphi(M)$, and extends in $(-\infty, \infty)$.

Now suppose y is a limiting point of $\varphi(M)$ in N, then $\exists\, x \in M$ and $\exists$ a geodesic $\gamma(t)$, $t \in [0,1]$, in N such that $\gamma(0) = \varphi(x)$ and $\gamma(1) = y$.

Therefore, by the above argument $y = \gamma(1) \subset \varphi(M)$

$\therefore$ $\varphi(M)$ is closed in N.

Hence $\varphi(M)$ is both open and closed (non-empty) in a connected mfd $N \Rightarrow$ we have $\varphi(M) = N$.

$\Rightarrow$ φ is <u>surjective</u>.

Note that, we've in fact proved the following

- commutative diagram:

$$\begin{tikzcd} T_xM \arrow[r, "d\varphi"] \arrow[d, "\exp_x^M"'] \arrow[dr, phantom, "\circlearrowright"] & T_{\varphi(x)}N \arrow[d, "\exp_{\varphi(x)}^N"] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

(local isom)

• and N is complete.

Even more: For $\delta > 0$ small s.t. $\exp_x$ is a diffeo. when restricted to a ball of radius δ, we have

$$\begin{tikzcd} B^M(\delta) \arrow[r, "d\varphi"] \arrow[d, "\exp^M_x"'] \arrow[dr, phantom, "\circlearrowright"] & B^N(\delta) \arrow[d, "\exp^N_{\varphi(x)}"] \\ B^M_\delta(x) \arrow[r, "\varphi"] & B^N_\delta(\varphi(x)) \end{tikzcd}$$

(Ex!)

(local isom)

Step 2: φ is a covering map.

We need to show that $\forall y \in N$, $\exists$ nbd U of y in N such that $\varphi^{-1}(U) = \bigcup_i W_i$ with

$$\begin{cases} W_i \cap W_j = \emptyset \text{ for } i \neq j \\ \varphi : W_i \to U \text{ is a diffeomorphism} \end{cases}$$

Pf of Step 2:

$\forall y \in N$, $\exists \delta > 0$ such that

$\exp_y^N : B^N(\delta) \to B_\delta^N$ is a diffeomaphism

where $B^N(\delta) = \{ v \in T_y N : |v|_N < \delta \}$

$B_\delta^N = \{ z \in N : d_N(z, y) < \delta \}$

Since φ is a local isom. & hence a local diffeo

$\varphi^{-1}(y)$ is a discrete set in M.

Let $\varphi^{-1}(y) = \{ x_i \}_{i \in \Lambda}$ for some index set Λ.

Denote
$$B^i(\delta) = B^M(x_i, \delta) = \{ v \in T_{x_i} M : |v|_M < \delta \}$$
$$B_\delta^i = B_\delta^M(x_i) = \{ \xi \in M : d_M(\xi, x_i) < \delta \}.$$

Claim: (i) $\varphi^{-1}(B_\delta^N) = \bigcup_i B_\delta^i$

(ii) $\forall i$, $\varphi : B_\delta^i \to B_\delta^N$ is a diffeo.

(iii) $\forall i \neq j$, $B_\delta^i \cap B_\delta^j = \phi$.

Pf of (i): It is clear that $\bigcup_i B_\delta^i \subset \varphi^{-1}(B_\delta^N)$

since φ is a local isom.

Conversely, fa $z \in \varphi^{-1}(B_\delta^N)$, we have

$$\varphi(z) \in B_\delta^N .$$

By the choice of $\delta > 0$, $\exists$ unique geodesic

$\gamma : [0,1] \to B_\delta^N$ such that

$$\gamma(0) = \varphi(z) \ \& \ \gamma(1) = y .$$

Then by the argument in the proof of Step 1,

$\exists$ a geodesic $\tilde{\gamma} : [0,1] \to M$ such that

$$\tilde{\gamma}(0) = z \ \& \ \varphi \circ \tilde{\gamma}(t) = \gamma(t), \ \forall t \in [0,1]$$

$$\Rightarrow \quad \varphi(\tilde{\gamma}(1)) = \gamma(1) = y$$

$$\Rightarrow \quad \tilde{\gamma}(1) \in \varphi^{-1}(y) = \{x_i\}_{i \in \Lambda}$$

$$\Rightarrow \quad \tilde{\gamma}(1) = x_i \text{ fa some } i \in \Lambda .$$

Again, using φ = local isom, we have

$$\text{Length}_M(\tilde{\gamma}) = \text{Length}_N(\gamma) < \delta$$

$\Rightarrow \quad \tilde{\gamma}(0) = z$ has a distance $< \delta$ to x_i

$\therefore \quad z \in B_\delta^i \subset \bigcup_i B_\delta^i$. This proves (i). $\boxtimes$