§5.2 Geodesic & Curvatures

Let $\mathbb{H}^n = \left(\mathbb{B}^n, \frac{4}{(1-|x|^2)^2} \sum_i dx^i \otimes dx^i\right)$

Facts: $\mathbb{R}^2 \hookrightarrow \mathbb{R}^n$, $S^2 \hookrightarrow S^n$, $\mathbb{H}^2 \hookrightarrow \mathbb{H}^n$

are totally geodesic submanifolds, the studies of geodesics on $\mathbb{R}^n$, S^n & $\mathbb{H}^n$ can be reduced to $\mathbb{R}^2$, S^2, & $\mathbb{H}^2$.

Let $M = \mathbb{R}^2, S^2$, or $\mathbb{H}^2$, and let $0 \in M$ be a fixed point.

Let $C(r) = \{x \in M : d(0,x) = r\}$ be the geodesic circle of radius r.

If $r > 0$, small enough, then

$$C(r) = \exp_0(\text{circle of radius } r \text{ in } T_0M)$$

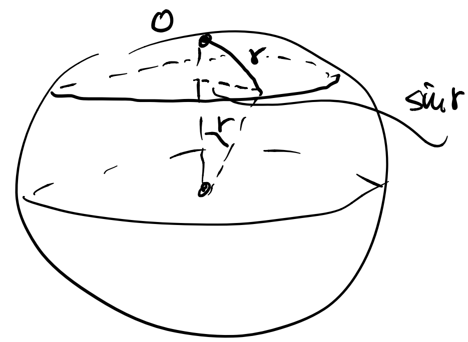
Denote

$$\text{length } C(r) = \begin{cases} C_0(r) & , \text{ if } M = \mathbb{R}^2 \\ C_+(r) & , \text{ if } M = S^2 \\ C_-(r) & , \text{ if } M = \mathbb{H}^2 \end{cases}$$

If $M = \mathbb{R}^2$, it is clear that

$$\boxed{C_0(r) = 2\pi r}$$

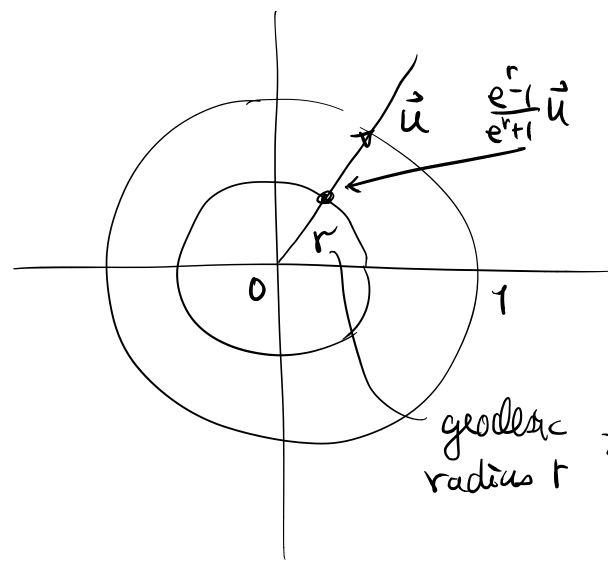
If $M = S^2$, we may assume O = North pole. Then it is easy see that the geodesic circle



$C(r)$ = a circle of radius $\sin r$ in $\mathbb{R}^3$

$$\Rightarrow \quad \boxed{C_+(r) = 2\pi \sin r} \qquad (\text{for small } r>0)$$

If $M = \mathbb{H}^2$, then by the proof of Lemma 6, a normalized geodesic from O is given by



$$\gamma(s) = \frac{e^s-1}{e^s+1}\vec{u},$$

where $\vec{u}$ = unit vector in $\mathbb{R}^2$,

s = arc-length $\left(|\gamma'(s)|_{\mathbb{H}^2} = 1 \right)$

$$\Rightarrow \quad d_{\mathbb{H}^2}(0, \gamma(r)) = \int_0^r |\gamma'(s)|_{\mathbb{H}^2} ds = r.$$

$\Rightarrow$ $C(r)$ = Euclidean circle of radius $\dfrac{e^r-1}{e^r+1}$ $\left(= \tanh\frac{r}{2}\right)$

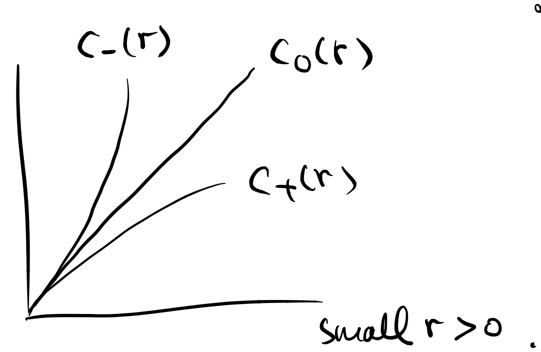
$$\Rightarrow \quad C_-(r) = \int_0^{2\pi} \frac{2}{1-\rho^2}\,\rho\, d\theta \qquad \text{where } \rho = \tanh\frac{r}{2}$$

$$= 2\pi \cdot \frac{2\rho}{1-\rho^2}$$

$$\Rightarrow \quad \boxed{C_-(r) = 2\pi \sinh r}$$

In summary, we have

$$\begin{cases} C_0(r) = 2\pi r \\ C_+(r) = 2\pi \sin r \\ C_-(r) = 2\pi \sinh r \end{cases}$$



To generalize the above to arbitrary <u>complete</u> Riem. manifold, we need to study <u>variations</u> of geodesics.

Let $\gamma : [a,b]\times[c,d] \to M$ be a C^∞ map from the rectangle $[a,b]\times[c,d]$ to a complete Riem. manifold M ($\dim \geq 2$).

Denote a point in $[a,b]\times[c,d]$ by (t,u).

Then we can define 2 tangent vector fields along γ

by
$$\begin{cases} T(t,u) = d\gamma\left(\frac{\partial}{\partial t}\Big|_{(t,u)}\right) \\ U(t,u) = d\gamma\left(\frac{\partial}{\partial u}\Big|_{(t,u)}\right) \end{cases}$$

$\forall$ fixed $u \in [c,d]$, a curve
$$\begin{array}{rcl} \gamma_u : [a,b] & \to & M \\ \Psi & & \Psi \\ t & \mapsto & \gamma(t,u) \end{array}$$
is defined.

Suppose $0 \in [c,d]$. Then γ_0 is called the <u>base curve</u> of γ. If γ_u are geodesics $\forall u \in [c,d]$, we call γ a <u>one-parameter family of geodesics</u>.

In this case, the vector field $T = \gamma_u'$ and hence
$$D_T T = 0.$$
We also have $[T, U] = d\gamma\left(\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial u}\right]\right) = 0$

Hence $\begin{cases} [T, U] = 0 \\ D_T T = 0 \end{cases}$ along γ.

This implies

$$R_{TU}T = -D_T(D_U T) + D_U(\cancel{D_T T}) + D_{\cancel{[T,U]}} T$$

$$= -D_T(D_T U) \quad \left(\begin{array}{l}\text{by Torsion free condition} \\ \Rightarrow D_T U = D_U T\end{array}\right)$$

Therefore, along the base geodesic γ_0, we have

$$\boxed{D_{\gamma_0'}(D_{\gamma_0'} U) + R_{\gamma_0' U}\gamma_0' = 0} \quad (\text{Jac})$$

or simply $\boxed{U'' + R_{\gamma_0' U}\gamma_0' = 0}$

where $U'' = D_{\gamma_0'}(D_{\gamma_0'} U)$ (similarly $U' = D_{\gamma_0'} U$)

Def: • Equation (Jac) is called the Jacobi equation along γ_0.

• Solutions of (Jac) are called Jacobi fields along γ_0.

Note: The vector field U constructed above is called a <u>transversal vector field</u> (or <u>variational vector field</u>) of $\{\gamma_u\}$.

Lemma 7: A transversal vector field of a 1-parameter family of geodesics is a Jacobi field.

Eg: If M = 2 dim'l complete Riem. manifold

Denote $C'(r) = \{x \in M : d(x, o) = r\}$

$c(r) = \text{length } C'(r)$

where $o \in M$ is fixed.

Let (ρ, θ) = polar coordinates on T_oM.

Let $\delta > 0$ small s.t. $\exp_o$ is a diffeomorphism on

$$B(\delta) = \{v \in T_oM : \rho(v) < \delta\}.$$

We can parametrize a circle of radius r in $B(\delta)$ by

$$\tilde{r} : [0, 2\pi] \to B(\delta)$$
$$\theta \mapsto (r, \theta)$$

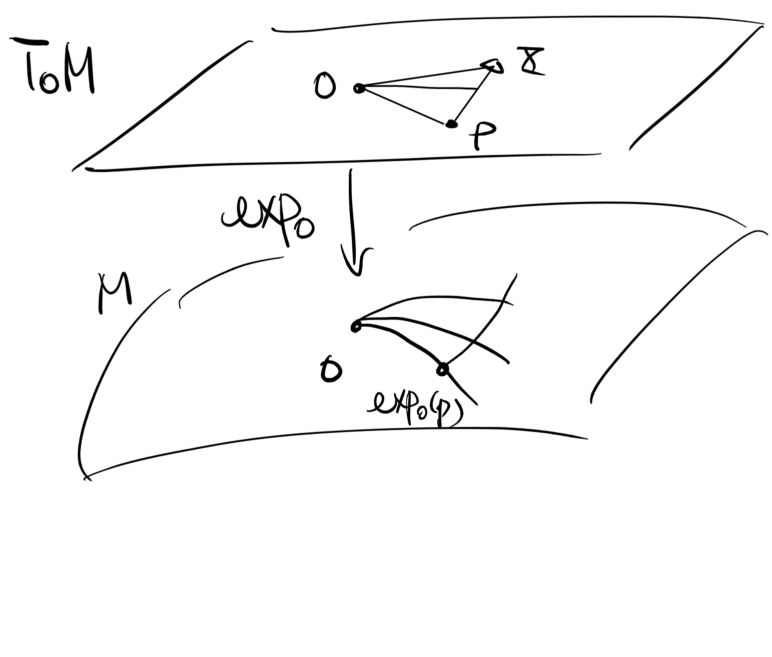
Then $C'(r) = \exp_o(\tilde{r})$ and

$$L(r) = \int_0^{2\pi} \left| (d\exp_o)_{(r,\theta)} \left(\frac{\partial}{\partial\theta}\right) \right| d\theta .$$

Note that $(d\exp_o)_{(r,\theta)}\left(\frac{\partial}{\partial\theta}\right)$ is a transversal vector field of the family of radial geodesics (with <u>specific initial values</u>)

<u>General setting</u>

Let
- M = complete Riem manifold of dim $n \geq 2$
- $O \in M$ fixed point
- $p \in T_oM$
- $X \in T_p(T_oM) \cong T_oM$



Define $\Gamma : [0,r] \times [0,1] \to M$, where $r = |p|$ by

$$\Gamma(t,u) = \exp_0 \left[\frac{t}{r}(p + uX) \right]$$

Then $\forall\, u \in [0,1]$, $\Gamma_u(t) = \Gamma(t,u)$ is a geodesic with initial tangent vector $\frac{1}{r}(p + uX)$ (not of length 1, unless $u = 0$) $\Rightarrow$

$\Gamma(t,u)$ is a 1-para. family of geodesics.

Let $V(t)$ = transversal vector field along Γ_0, and

$\delta > 0$ be a number s.t. $\exp_0$ is a diffeo on

$B(\delta) = \{ v \in T_0M : |v| < \delta \}$ ($|v| = \rho(v)$ in polar)

Set $B_\delta = \{ x \in M : d(0,x) < \delta \}$. Then

$$B_\delta = \exp_0(B(\delta))$$

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of T_0M and

$\{\alpha^1, \cdots, \alpha^n\}$ be the dual basis of $\{e_1, \cdots, e_n\}$.

Then $\{\alpha^1, \cdots, \alpha^n\}$ are coordinate functions on T_0M.

Define a coordinate system on B_δ by

$$x^i = \alpha^i \circ \exp_0^{-1} : B_\delta \to \mathbb{R} \quad (i=1,\cdots,n)$$

Then

Claim: $\begin{cases} \left\langle \frac{\partial}{\partial x^i}\Big|_0, \frac{\partial}{\partial x^j}\Big|_0 \right\rangle = \delta_{ij}, & \forall i,j \\ D_{\frac{\partial}{\partial x^i}|_0} \frac{\partial}{\partial x^j} = 0, & \forall i,j \end{cases}$

Pf: The 1^{st} eqt. follows from $(d\exp_{\underset{0 \in M}{0}})_{\underset{0 \in T_0M}{0}} = Id$

To see the 2^{nd}, we define a bilinear form

$$\beta : T_0M \times T_0M \to \mathbb{R}^n$$

by $\quad \beta(e_i, e_j) = D_{\frac{\partial}{\partial x^i}|_0} \frac{\partial}{\partial x^j}$.

Then $\forall\, v = \sum v^i e_i \in T_0M$

$$\beta(v,v) = \sum_{i,j} v^i v^j \beta(e_i, e_j) = \sum_{i,j} v^i v^j D_{\frac{\partial}{\partial x^i}|_0} \frac{\partial}{\partial x^j}$$

$$= D_{\left(\sum_i v^i \frac{\partial}{\partial x^i}|_0\right)} \left(\sum_j v^j \frac{\partial}{\partial x^j}\right)$$

Note that $\sum v^i \frac{\partial}{\partial x^i}\Big|_0$ is the initial tangent vector of the geodesic $\exp_0\left(t \sum_i v^i e_i\right)$. Hence $\beta(v,v)=0$ by the geodesic eqt. $\Rightarrow$ $\beta(v,v)\equiv 0$ $\forall\, v\in T_0M$

$$\therefore \quad D_{\frac{\partial}{\partial x^i}\big|_0} \frac{\partial}{\partial x^j} = 0 \,, \quad \forall\, i,j. \qquad ※$$

Note: coordinate systems satisfying these conditions are called normal coordinate systems.

Now assume $p=\sum p^i e_i$ & $X=\sum X^i e_i$ (under $T_p(T_0M) \simeq T_0M$)

For $\varepsilon>0$, small, εp & $\varepsilon X \in B(\delta)$

Then in the above coordinate system $\{x^1,\cdots,x^n\}$, the coordinate vector of $\Gamma(t,u)=\exp_0\left(\frac{t}{r}(p+uX)\right)$ is

$$\frac{t}{r}(\vec{p}+u\vec{X}) \,, \text{ where } \vec{p}=\begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} \text{ \& } \vec{X}=\begin{pmatrix} X^1 \\ \vdots \\ X^n \end{pmatrix}$$

for $(t,u)\in[0,\varepsilon r]\times[0,\varepsilon]$.

And the base geodesic is $\Gamma_0(t)=\Gamma(t,0) \overset{\text{in coordinates}}{=} \frac{t}{r}\vec{p}$

$$\Rightarrow \quad U(t) = \frac{\partial}{\partial u}\Gamma(t,u) = \frac{t}{r}\vec{X} \quad (\text{in coordinates})$$

i.e. $$U(t) = \frac{t}{r}\sum X^i \frac{\partial}{\partial x^i}\Big|_{(t,0)}$$

Therefore $U(0)=0$ and

$$U'(0) = D_{\Gamma_0'(0)} U = \frac{d}{dt}\Big|_{t=0}\left(\frac{t}{r}\sum X^i \frac{\partial}{\partial x^i}\Big|_{(t,0)}\right)$$

$$= \frac{1}{r}\sum X^i \frac{\partial}{\partial x^i}\Big|_0 + 0$$

In conclusion, the transversal vector field $U(t)$ of $\Gamma(t,u) = \exp_0\left[\frac{t}{r}(p+uX)\right]$ satisfies

$$\begin{cases} U(0)=0 \\ U'(0) = \frac{1}{r}\vec{X} \quad (\text{in coordinates}), \text{ where } r=|p| \end{cases}$$

$$\left[\underline{\text{Check}}: \; U(t) = \frac{t}{r}(d\exp_0)_{(\frac{t}{r}p)}(X)\right]$$

Applying the above to $M = \mathbb{R}^2, S^2$ or $\mathbb{H}^2$ with $p=(r,\theta)$, $X = \frac{\partial}{\partial\theta}\Big|_{(r,\theta)}$.

Then $V(r) = (d\exp_0)_{(r,\theta)}\left(\frac{\partial}{\partial\theta}\right)$ (at $t=r$)

is a Jacobi field satisfying

$$\begin{cases} V(0)=0 \\ |V'(0)| = \frac{1}{r}\left|\frac{\partial}{\partial\theta}\right| = 1 \end{cases} \qquad ((r,\theta) = \text{polar})$$

Let $W(t)$ = <u>unit</u> parallel vector field along Γ_0

s.t.

$$\langle W(t), \Gamma_0'(t)\rangle = 0.$$

By Gauss Lemma

$\Rightarrow$ $V(t) = (d\exp_0)_{(t,\theta)}\left(\frac{\partial}{\partial\theta}\right)$ is normal

to $\Gamma_0'(t)$.

In our case of dim $=2$,

$$V(t) = (d\exp_0)_{(t,\theta)}\left(\frac{\partial}{\partial\theta}\right) = f(t)\,W(t)$$

for some function $f \in C^\infty[0,r]$.

Then $V'(t) = D_{\Gamma_0'(t)} V(t) = f'(t)\,W(t)$

$$V''(t) = D_{\Gamma_0'(t)} D_{\Gamma_0'(t)} V(t) = f''(t)\, W(t)$$

(since W is parallel)

$$(\text{Jac}) \Rightarrow \quad f''(t)W(t) + R_{\Gamma_0', fW}\Gamma_0' = 0$$

$$\Rightarrow \quad f''(t) + f\, \langle R_{\Gamma_0' W}\Gamma_0', W \rangle = 0$$

i.e. $f'' + Kf = 0$, where $K =$ Gauss curvature at $\Gamma_0(t)$

(in general, $K =$ sectional curvature ($\operatorname{span}\langle \Gamma_0', W\rangle$)
since $|\Gamma_0'(t)| = |W(t)| = 1$, & $\langle \Gamma_0', W\rangle = 0$)

We may also assume $\langle W, \frac{\partial}{\partial\theta}\rangle > 0$, we have

$$\begin{cases} f'' + Kf = 0 \\ f(0) = 0 \\ f'(0) = 1 \end{cases}$$

$\therefore$ The signature of K has implication on

$$c(r) = \int_0^{2\pi} \left| (d\exp_p)_{(r,\theta)} \left(\tfrac{\partial}{\partial\theta}\right) \right| d\theta = \int_0^{2\pi} f\, d\theta$$

Particular cases: $K \equiv 0, \pm 1$, we have

$$f(r) = \begin{cases} r & , K \equiv 0 \\ \sin r & , K \equiv +1 \\ \sinh r & , K \equiv -1 . \end{cases}$$

Prop: Let $K \geq +1$, then $C(r) \leq 2\pi \sin r$, for small r.

Pf: Consider a comparison function $h(t) = \sin t$

Then $\begin{cases} h'' + h = 0 \\ h(0) = 0 \\ h'(0) = 1 \end{cases}$

$$\Rightarrow \quad (hf' - fh')' = hf'' - fh''$$
$$= -Kfh + fh$$
$$= -(K-1) fh$$

Since $f(0) = h(0) = 0$, $f'(0) = h'(0) = 1$, we have

$$f \geq 0, \ h \geq 0 \quad \text{for small } t > 0$$

$$\Rightarrow \quad (hf' - fh')' \leq 0 \quad \text{for small } t > 0 .$$

$$\Rightarrow \quad hf' - fh' \leq h(0)f'(0) - f(0)h'(0) , \quad \text{for small } t > 0.$$

$$= 0$$

$$\Rightarrow \quad \left(\frac{f}{h}\right)' = \frac{hf' - fh'}{h^2} \leq 0 \qquad \text{for small } t > 0$$

$$\Rightarrow \quad \frac{f}{h}(t) \leq \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{h(\varepsilon)} = \frac{f'(0)}{h'(0)} = 1$$

for small $t > 0$

$$\Rightarrow \quad f(t) \leq h(t) = \sin t \quad \text{for small } t > 0$$

Since the about estimate is indep. of θ,

hence $\quad C(r) = \int_0^{2\pi} f(r,\theta)\,d\theta \leq 2\pi \sin r$ for small $r > 0$

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Prop: If $K \leq -1$, we have $C(r) \geq 2\pi \sinh r$

(for small $r > 0$ at this moment)

Pf: Consider $h(t) = \sinh t$

Then
$$\begin{cases} h'' - h = 0 \\ h(0) = 0 \\ h'(0) = 1 \end{cases}$$

$$\Rightarrow (hf' - fh')' = hf'' - fh'' = -Kfh - fh$$

$$= -(K+1)fh \geq 0 \quad \text{for small } t > 0.$$

$$\Rightarrow \quad hf' - fh' \geq h(0)f'(0) - f(0)h'(0) = 0$$

$$\Rightarrow \quad \left(\frac{f}{h}\right)' \geq 0$$

$$\Rightarrow \quad \frac{f}{h}(t) \geq \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{h(\varepsilon)} = \frac{f'(0)}{h'(0)} = 1 .$$

$$\Rightarrow \quad f(t) \geq \sinh(t) \quad \text{for small } t > 0.$$

$$\Rightarrow \quad C(r) \geq 2\pi \sinh(r) \quad , \text{for small } r > 0 .$$

#

Ch6 Jacobi Field, Cartan-Hadamard Theorem

§6.1 Jacobi Field

Let γ = normalized geodesic (i.e. $|\gamma'|=1$)

Recall that the Jacobi equation (for vector field along γ) is

$$\boxed{U'' + R_{\gamma' U}\gamma' = 0} \quad (\text{Jac})$$

where $U'' = D_{\gamma'} D_{\gamma'} U$ $\quad (U' = D_{\gamma'} U)$.

Let $\{e_1(t), \cdots, e_n(t)\}$ be parallel vector fields along γ such that $\forall t$

$$\begin{cases} e_1(t) = \gamma'(t) \\ \{e_i(t)\}_{i=1}^{n} \text{ is an orthonormal basis of } T_{\gamma(t)}M \end{cases}$$

Then $\forall$ vector field U along γ, we write

$$U(t) = \sum_i f^i(t) e_i(t) \quad , \text{ for some functions } f^i(t).$$

Similarly, the curvature can be written as

$$R_{e_i(t)\, e_j(t)}\, e_k(t) = \sum_l R_{ijk}^{l}\, e_l(t)$$

where $R_{ijk}^{l}(t) = \langle R_{e_i(t) e_j(t)} e_k(t), e_l(t) \rangle$.

Then the eqt. (Jac) $\Rightarrow$

$$
\begin{aligned}
0 &= V'' + R_{\gamma' v}\gamma' \\
&= \left(\sum f^i e_i\right)'' + R_{e_1 (\sum f^i e_i)} e_1 \\
&= \sum_i (f^i)'' e_i + \sum_l f^l R_{e_1 e_l} e_1 \\
&= \sum_i (f^i)'' e_i + \sum_l f^l \left(\sum_i R^i_{1l1} e_i\right) \\
&= \sum_i \left[(f^i)'' + \sum_l f^l R^i_{1l1}\right] e_i
\end{aligned}
$$

$\therefore$ (Jac) $\Leftrightarrow$ $\boxed{(f^i)'' + \sum_l R^i_{1l1} f^l = 0, \quad \forall i}$

which is a 2^{nd} order linear ODE system.

Then ODE theory $\Rightarrow$

<u>Lemma 1</u>

(i) Let γ be a geodesic. Then given any $v, w \in T_{\gamma(0)}M$ $\exists$ unique Jacobi field $V(t)$ along γ s.t.

$$U(0) = v, \quad U'(0) = w.$$

(2) Unless $U \equiv 0$, the zero set of $U(t)$ along γ is discrete.

In fact, we have

<u>Lemma 2</u>: Let U be a vector field along a normalized geodesic γ. Then

U is a Jacobi field along γ

$\Leftrightarrow$ U is a transversal vector field of a one-parameter family of geodesics.

<u>Pf of lemma 2</u>:

($\Leftarrow$) Proved in previous chapter.