Remark: The formula

$$(D_v \rho)(X) = v(\rho(X)) - \rho(D_v X), \ \forall X \in \Gamma(M)$$

shows that $D_v K$ doesn't depend on the representative γ of v.

Def: Let K = tensor field on M

X = vector field on M

Then we define $(D_X K)(x) \overset{\text{def}}{=\!=} D_{X(x)} K$, $\forall x \in M$.

Note: By linearity of $D_X K$ in X, we can define

$$DK \in (\otimes^r TM) \otimes (\otimes^{s+1} T^*M)$$

(for $K \in (\otimes^r TM) \otimes (\otimes^s T^*M)$) by requiring

$$(DK)(\omega^1 \otimes \cdots \otimes \omega^r \otimes X_1 \otimes \cdots \otimes X_s \otimes X)$$

$$\overset{\text{def}}{=\!=} (D_X K)(\omega^1 \otimes \cdots \otimes \omega^r \otimes X_1 \otimes \cdots \otimes X_s)$$

[Caution: Some authors put
$(DK)(\omega^1 \otimes \cdots \otimes \omega^r \otimes X \otimes X_1 \otimes \cdots \otimes X_s) = (D_X K)(\omega^1 \otimes \cdots \otimes \omega^r \otimes X_1 \otimes \cdots \otimes X_s)$]

Note: If $K = f \in T^{(0,0)}M \cong C^\infty(M)$.

Then $Df = df$ = the usual differential of f

(Ex!)

Def: For $n \geq 0$, we define

$$D^{n+1}K = D(D^n K)$$

Note: $(D^2K)(\cdots, X, Y) \neq D^2K(\cdots, Y, X)$ in general.

eg: Let $K = f \in C^\infty(M)$

Then
$$\begin{aligned}(D^2 f)(X, Y) &= (D\,df)(X, Y) \\ &= (D_Y df)(X) \\ &= Y(df(X)) - df(D_Y X) \\ &= Y(Xf) - df(D_Y X) \\ &\neq D_Y(D_X f)\end{aligned}$$

(by definition $D_Y(D_X f) = D_Y(X(f)) = Y(X(f))$)

Similarly $(D^2 f)(Y, X) = X(Yf) - df(D_X Y)$

$\Rightarrow (D^2 f)(X, Y) - (D^2 f)(Y, X)$

$$= Y(Xf) - X(Yf) - (D_Y X)(f) + (D_X Y)(f)$$

$$= \left(-[X,Y] + D_X Y - D_Y X\right)(f)$$

$$= \underbrace{T(X,Y)}_{\text{torsion tensor}} f$$

$\therefore$ D symmetric (torsion free) $\Leftrightarrow$ $D^2 f$ is symmetric

In this case, $D^2 f$ is called the <u>Hessian</u> of f.

<u>From now on</u>, we assume M has a Riemannian metric g and $D =$ Levi-Civita connection of g. Therefore, $D^2 f$ is always symmetric for $f \in C^\infty(M)$.

<u>Def</u>: $\forall$ symmetric $S \in \otimes^2 T^*M$, we define $\mathrm{tr}_g S \in C^\infty(M)$ the <u>trace of S</u>, by

$$\mathrm{tr}_g S(x) = \sum_i S(e_i, e_i)$$

where $\{e_i\}$ is an <u>orthonormal</u> basis of $T_x M$.

Ex: Check: (i) $\mathrm{tr}_g S$ is well-defined, i.e. independent of the choice of $\{e_i\}$.

(ii) $tr_{,}S'(x)$ is smooth in x.

Def: Let (M,g) = Riemannian manifold

D = Levi-Civita connection of g.

Then the <u>Laplace operator</u>, <u>Laplacian</u>, or <u>Laplace-Beltrami operator</u>

$$\Delta : C^\infty(M) \to C^\infty(M)$$

is defined by $\Delta f = tr\, D^2 f$.

Ex: Prove that in local coordinates $(x^1, \cdots, x^n)$

$$\Delta f = \frac{1}{\sqrt{G}} \sum_j \frac{\partial}{\partial x^j}\left(\sum_i g^{ij} \sqrt{G} \frac{\partial f}{\partial x^i} \right)$$

where $G = \det(g_{ij})$, $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ and

$(g^{ij}) = (g_{ij})^{-1}$.

3.2 Curvature Tensor

Let $\mathcal{T}^* =$ Algebra of tensor fields on M / $C^\infty(M)$.

Then $\forall$ vector field $X \in \Gamma(M)$

$$D_X : \mathcal{T}^* \to \mathcal{T}^* \text{ is a derivation.}$$

Therefore, if we have D_X & D_Y, the Lie bracket

$$[D_X, D_Y] = D_X D_Y - D_Y D_X$$

is also a derivation (Ex!)

Hence we can make the following definition

$$\boxed{\begin{aligned} R_{XY} &= D_{[X,Y]} - [D_X, D_Y] \\ &= -D_X D_Y + D_Y D_X + D_{[X,Y]} \end{aligned}}$$

Prop :

(1) $R_{XY} : \mathcal{T}^* \to \mathcal{T}^*$ is a derivation

(2) R_{XY} preserves the type of a tensor field

i.e. $K = (r,s)$-type $\Rightarrow$ $R_{XY}K = (r,s)$-type.

(3) $\forall f \in C^\infty(M)$,

$$R_{(fX)Y}K = R_{X(fY)}K = R_{XY}(fK) = f R_{XY}K$$

(4) $\forall f \in C^\infty(M)$, $R_{XY} f = 0$.

Pf: We check only $R_{(fX)Y}K = f R_{XY}K$

(the others are similar & easy ex!)

$$\begin{aligned}
R_{(fX)Y}K &= -D_{fX}D_Y K + D_Y D_{fX}K + D_{[fX,Y]}K \\
&= -f D_X D_Y K + D_Y(f D_X K) + D_{[fX,Y]}K \\
&= -f D_X D_Y K + f D_Y D_X K + (Yf) D_X K + D_{[fX,Y]}K \\
&= f\left(-D_X D_Y K + D_Y D_X K + D_{[X,Y]}K\right) \\
&\quad - f D_{[X,Y]}K + (Yf) D_X K + D_{[fX,Y]}K \\
&= f R_{XY}K + D_{(\underbrace{[fX,Y] - f[X,Y] + (Yf)X}_{=0})}K \\
&= f R_{XY}K. \quad \#
\end{aligned}$$

($\therefore$ $D_{[X,Y]}$ is needed in the definition in order to have property (3))

Note: By property (3), if $K = Z$ is a vector field, then one can use $R_{XY}Z$ to define a (1,3)-tensor

$$(\omega, X, Y, Z) \xrightarrow{R} \omega(R_{XY}Z)$$

($\forall$ 1-form ω & vector fields X, Y, Z)

It also defines a $(0,4)$-tensor R (using metric g)

$$R(X,Y,Z,W) = g(R_{XY}Z, W)$$

<u>Def</u>: $R_{XY}Z$ or $R(X,Y,Z,W)$ are called the (Riemannian) <u>curvature tensor</u> of g. (More precisely, R is the curvature tensor of g)

<u>Local formula</u>: In a coordinate system $(x^1, \cdots, x^n)$,

if $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right) \quad \text{(Christoffel symbol)}$$

then $R_{ijkl} \overset{\text{def}}{=\!=} R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l})$

is given by

$$\boxed{\begin{aligned} R_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}\right) \\ + (g_{rs}\Gamma^r_{jk}\Gamma^s_{il} + g_{rs}\Gamma^r_{jl}\Gamma^s_{ik}) \end{aligned}} \quad \text{(Pf: Ex!)}$$

Note: (i) $R = R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l$

(ii) R is a 2nd order non-linear function of g.

Def: Let (M, g) & (N, h) be 2 Riemannian manifolds.

A C^∞ map $\varphi: M \to N$ is called a local isometry

$\Leftrightarrow \forall x \in M$,

$$d\varphi: (T_x M, g_x) \to (T_{\varphi(x)} N, h_{\varphi(x)})$$

is an isometry of the inner product spaces.

i.e. $\forall v_1, v_2 \in T_x M$

$$\boxed{h_{\varphi(x)}(d\varphi(v_1), d\varphi(v_2)) = g_x(v_1, v_2)}$$

Note: If φ = local isom, then $\dim M = \dim N$.

and φ is an immersion.

Def: $\varphi: (M, g) \to (N, h)$ is called a global isometry

or simply an isometry,

$\Leftrightarrow$ φ is a local isometry and a diffeomorphism.

Fact: Let $\varphi: (M, g) \to (M', g')$ is an isometry

- D = Levi-Civita connection of g
- D' = Levi-Civita connection of g'
- $X, Y \in \Gamma(M)$ and
  $d\varphi(X) = X'$, $d\varphi(Y) = Y' \in \Gamma(M')$

Then $$\boxed{d\varphi(D_X Y) = D'_{X'} Y'}$$

i.e. Levi-Civita connection is a metric invariant.

Thm (Metric invariance of curvature tensor)

Let
- $\varphi : (M, g) \to (M', g')$ is an isometry
- R, R' = curvature tensors of g & g' respectively,
- $X, Y, Z, W \in \Gamma(M)$ &
  $X' = d\varphi(X)$, $Y' = d\varphi(Y)$, $Z' = d\varphi(Z)$, $W' = d\varphi(W) \in \Gamma(M')$

Then $$\boxed{\begin{aligned} d\varphi(R_{XY}Z) &= R'_{X'Y'}Z' \\ R(X,Y,Z,W) &= R'(X',Y',Z',W')\circ\varphi \end{aligned}}$$

(Pf: Ex!)

Note: If $\dim M = 2$, then one can define the Gaussian curvature $K : M \to \mathbb{R}$ by

$$K(x) = R(e_1, e_2, e_1, e_2)(x), \quad \forall x \in M$$

for any orthonormal basis $\{e_1, e_2\}$ of T_xM

And this K coincides with the original definition for $M^2 \subset \mathbb{R}^3$.

Def: A Riemannian manifold (M, g) is called flat if its curvature tensor $R \equiv 0$.

eg: $(\mathbb{R}^n, \text{standard metric}) = (\mathbb{R}^n, dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n)$ is flat (Reason: $g_{ij} = \text{const} \Rightarrow \Gamma_{ij}^k = 0 \Rightarrow R = 0$)

3.3 Basic properties of curvature tensor

Lemma 1 $\forall$ vector fields X, Y, Z, W

(1) $R_{XY} = -R_{YX}$

(2) (1st Bianchi identity)

$$R_{XY}Z + R_{YZ}X + R_{ZX}Y = 0$$

(3) $R(X, Y, Z, W) = -R(X, Y, W, Z)$

(4) $R(X, Y, Z, W) = R(Z, W, X, Y)$

Pf: (1) is clear.

For (2) & (3), we only need to check the case that $\{X, Y, Z, W\} = \{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\}$ (Since R is a tensor)

In this case, $0 = [X, Y] = \cdots = [Z, W]$

Hence
$$\begin{cases} D_X Y = D_Y X \\ R_{XY} = -D_X D_Y + D_Y D_X \end{cases}$$

$$\Rightarrow R_{XY}Z + R_{YZ}X + R_{ZX}Y$$
$$= (-D_X D_Y Z + D_Y D_X Z) + (-D_Y D_Z X + D_Z D_Y X) + (-D_Z D_X Y + D_X D_Z Y)$$

$$= D_X(\cancel{D_Z Y - D_Y Z}) + D_Y(\cancel{D_X Z - D_Z X}) + D_Z(\cancel{D_Y X - D_X Y})$$

$$= 0$$

This proves (2).

For (3), we 1st look at

$$R(X,Y,Z,Z) = \langle R_{XY}Z, Z\rangle$$

$$= \langle -D_X D_Y Z + D_Y D_X Z, Z\rangle$$

$$= -X\langle D_Y Z, Z\rangle + \langle D_Y Z, D_X Z\rangle$$

$$+ Y\langle D_X Z, Z\rangle - \langle D_X Z, D_Y Z\rangle$$

$$= -\frac{1}{2} X(Y\langle Z,Z\rangle) + \frac{1}{2} Y(X\langle Z,Z\rangle)$$

$$= -\frac{1}{2}\cancel{[X,Y]}\langle Z,Z\rangle = 0.$$

$$\Rightarrow \quad 0 = R(X,Y,Z+W,Z+W)$$

$$= \cancel{R(X,Y,Z,Z)} + R(X,Y,Z,W) + R(X,Y,W,Z)$$

$$+ \cancel{R(X,Y,W,W)}$$

$$\Rightarrow \quad R(X,Y,Z,W) = -R(X,Y,W,Z).$$ This proves (3).

<u>Proof of (4)</u> (Jost)

$$R(X,Y,Z,W) = -R(Y,X,Z,W) \qquad (\text{by } (1))$$

$$= R(Z,Y,X,W) + R(X,Z,Y,W) \quad (\text{by } (2)) \quad (1^{st} \text{ Bianchi})$$

Similarly

$$R(X,Y,Z,W) = -R(X,Y,W,Z) \quad (\text{by } (3))$$

$$= R(Y,W,X,Z) + R(W,X,Y,Z) \quad \left(\text{by } 1^{st} \text{ Bianchi}\right)$$

$\Rightarrow$

$$2R(X,Y,Z,W) = R(Z,Y,X,W) + R(X,Z,Y,W) + R(Y,W,X,Z) + R(W,X,Y,Z) \quad \text{— } (*)$$

Similarly

$$2R(Z,W,X,Y) = R(X,W,Z,Y) + R(Z,X,W,Y) + R(W,Y,Z,X) + R(Y,Z,W,X)$$

$$(\text{by } (1) \,\&\, (3)) = R(W,X,Y,Z) + R(X,Z,Y,W) + R(Y,W,X,Z) + R(Z,Y,X,W)$$

$$(\text{by } (*)) = 2R(X,Y,Z,W)$$

$\divideontimes$

<u>Lemma 2</u> Let $Q(X,Y) \stackrel{\text{def}}{=\!=} R(X,Y,X,Y)$

Then Q determines R

i.e. If R, R' are tensor fields satisfying (1)–(4) in <u>Lemma 1</u>, then $Q = Q' \Rightarrow R = R'$.

(Pf: Omitted)

<u>Def</u>: Let • π be a 2-dimensional subspace in $T_x M$

• $\{v_1, v_2\}$ = basis of π

Then
$$\boxed{K(\pi) = \frac{R(v_1, v_2, v_1, v_2)}{|v_1 \wedge v_2|^2}}$$

where
$$|v_1 \wedge v_2|^2 = \det(\langle v_i, v_j\rangle) = |v_1|^2|v_2|^2 - \langle v_1, v_2\rangle^2,$$

is called the <u>sectional curvature</u> of π.

<u>Note</u>:
- $K(\pi)$ doesn't depend on the basis $\{v_1, v_2\}$.
- If $\{e_1, e_2\}$ = orthonormal basis of π, then
$$K(\pi) = R(e_1, e_2, e_1, e_2)$$
- Lemma 2 $\Rightarrow$ K determines R

- Sectional curvature K is a metric invariant

i.e. If $\varphi: M \to M'$ isometry, $\pi \in T_xM$, $\pi' \subset T_{\varphi(x)}M'$ are 2-dim'l subspace with $\pi' = d\varphi(\pi)$. Then

$$K(\pi) = K'(\pi')$$

eg: If $K(\pi)=0$, $\forall x$ & $\pi^2 \subset T_xM$, then $R \equiv 0$.

Lemma3 (The 2nd Bianchi Identity)

$$\boxed{(D_X R)_{YZ} + (D_Y R)_{ZX} + (D_Z R)_{XY} = 0 \quad \forall X, Y, Z \in \Gamma(TM)}$$

$$\left(\begin{array}{l} \text{i.e. } (D_X R)(Y,Z,\cdot,\cdot) + (D_Y R)(Z,X,\cdot,\cdot) + (D_Z R)(X,Y,\cdot,\cdot) = 0 \\ \text{or } (D_X R)_{YZ} W + (D_Y R)_{ZX} W + (D_Z R)_{XY} W = 0 \end{array} \right)$$

Pf: It is sufficient to prove the identity for vector fields satisfying $[X,Y] = \cdots = 0$.

For these vector fields
$$\begin{cases} D_X Y = D_Y X \\ R_{XY} = -D_X D_Y + D_Y D_X \end{cases}$$

By definition

$$(D_X R)_{YZ} W = D_X(R_{YZ}W) - R_{(D_X Y)Z} W - R_{Y(D_X Z)} W - R_{YZ}(D_X W)$$

$$(D_Y R)_{ZX} W = D_Y(R_{ZX}W) - R_{(D_Y Z)X} W - R_{Z(D_Y X)} W - R_{ZX}(D_Y W)$$

$$(D_Z R)_{XY} W = D_Z(R_{XY}W) - R_{(D_Z X)Y} W - R_{X(D_Z Y)} W - R_{XY}(D_Z W)$$

$$\Rightarrow (D_X R)_{YZ} W + (D_Y R)_{ZX} W + (D_Z R)_{XY} W$$

$$= D_X(\overset{1}{\cancel{-D_Y D_Z W}} + \overset{2}{\cancel{D_Z D_Y W}}) + D_Y(\overset{3}{\cancel{-D_Z D_X W}} + \overset{4}{\cancel{D_X D_Z W}})$$

$$+ D_Z(\overset{5}{\cancel{-D_X D_Y W}} + \overset{6}{\cancel{D_Y D_X W}}) - (\overset{3}{\cancel{-D_Y D_Z}} + \overset{6}{\cancel{D_Z D_Y}})(D_X W)$$

$$- (\overset{5}{\cancel{-D_Z D_X}} + \overset{2}{\cancel{D_X D_Z}})(D_Y W) - (\overset{1}{\cancel{-D_X D_Y}} + \overset{4}{\cancel{D_Y D_X}})(D_Z W)$$

$$\left(\cancel{-R_{(D_X Y)Z}W + R_{(D_Y X)Z}W}\right)\left(\cancel{-R_{Y(D_X Z)}W + R_{Y(D_Z X)}W}\right)$$

$$\left(\cancel{-R_{(D_Y Z)X}W + R_{(D_Z Y)X}W}\right)$$

$$= 0$$