Solution to Midterm 2

1. (a) By direct computation, we have the following.

$$\begin{aligned}\text{RHS} &= \|x+y\|^2 + \|x-y\|^2 \\ &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &\quad + \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle \\ &= 2\|x\|^2 + 2\|y\|^2 = \text{LHS}\end{aligned}$$

(b) Using the parallelogram law, we have the following.

$$\begin{aligned}2\|u\|^2 + 2\|v\|^2 &= \|u+v\|^2 + \|u-v\|^2 \\ 2(\sqrt{2})^2 + 2\|v\|^2 &= (4)^2 + (2)^2 \\ \|v\| &= 2\sqrt{2}\end{aligned}$$

2. (a) By applying the Gram-Schmidt process, we have the following.

$$\begin{aligned}v_1 &= 1 \\ v_2 &= x - \frac{\langle x,1\rangle}{\langle 1,1\rangle}\cdot 1 = x - \frac{1}{2} \\ v_3 &= x^2 - \frac{\langle x^2,1\rangle}{\langle 1,1\rangle}\cdot 1 - \frac{\langle x^2, x-\frac{1}{2}\rangle}{\langle x-\frac{1}{2}, x-\frac{1}{2}\rangle}\cdot\left(x-\frac{1}{2}\right) \\ &= x^2 - \frac{1}{3} - \frac{\frac{1}{12}}{\frac{1}{12}}\left(x-\frac{1}{2}\right) \\ &= x^2 - x + \frac{1}{6}\end{aligned}$$

Then we can normalize them to obtain an orthonormal basis.

$$\begin{aligned}w_1 &= \frac{v_1}{\|v_1\|} = 1 \\ w_2 &= \frac{v_2}{\|v_2\|} = 2\sqrt{3}\left(x-\frac{1}{2}\right) \\ w_3 &= \frac{v_3}{\|v_3\|} = 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right)\end{aligned}$$

Hence, we have $\beta' = \{1, 2\sqrt{3}(x-\frac{1}{2}), 6\sqrt{5}(x^2-x+\frac{1}{6})\}$.

(b) Note that $\beta = \{1, x, x^2\}$ is a basis consisting of eigenvectors of T.

$$T(1) = 0, \quad T(x) = x, \quad T(x^2) = 0$$

Hence, T is diagonalizable.

(c) From the above, we see that

$$\begin{aligned} T(w_1) &= 0 \\ T(w_2) &= 2\sqrt{3}x = \sqrt{3}w_1 + w_2 \\ T(w_3) &= -6\sqrt{5}x = -\sqrt{15}w_1 - \sqrt{15}w_2 \end{aligned}$$

Note that β' is an orthonormal basis for $P_2(\mathbb{R})$. However, we have

$$[T]_{\beta'} = \begin{pmatrix} 0 & \sqrt{3} & -\sqrt{15} \\ 0 & 1 & -\sqrt{15} \\ 0 & 0 & 0 \end{pmatrix},$$

which is not self-adjoint. So, T is not self-adjoint and there does not exist an orthonormal eigenbasis of $P_2(\mathbb{R})$ corresponding to T.

3. (a) For any $c \in \mathbb{F}$, we have the following.

$$\begin{aligned} T_{y,z}(x_1 + cx_2) &= \langle x_1 + cx_2, y\rangle z \\ &= \langle x_1, y\rangle z + c\langle x_2, y\rangle z \\ &= T_{y,z}(x_1) + cT_{y,z}(x_2) \end{aligned}$$

Hence, we see that $T_{y,z}$ is linear.

(b) For any $x \in V$, we have the following.

$$\begin{aligned} T_{w,v}T_{y,z}(x) &= T_{w,v}(\langle x, y\rangle z) \\ &= \langle (\langle x, y\rangle z), w\rangle v \\ &= \langle x, y\rangle \langle z, w\rangle v \qquad \text{(note that } \langle x, y\rangle \text{ is just a scalar)} \\ &= \langle x, y\rangle (\langle z, w\rangle v) \\ &= T_{y,\langle z,w\rangle v}(x) \end{aligned}$$

Hence, we have $T_{w,v}T_{y,z} = T_{y,\langle z,w\rangle v}$.

(c) Given $y, z \in V$, for any $w, x \in V$, we have the following.

$$\begin{aligned} \left\langle w, T^*_{y,z}(x)\right\rangle &= \langle T_{y,z}(w), x\rangle \\ &= \langle \langle w, y\rangle z, x\rangle \\ &= \langle w, y\rangle \langle z, x\rangle \qquad \text{(again, } \langle w, y\rangle \text{ is just a scalar)} \\ &= \left\langle w, \overline{\langle z, x\rangle} y\right\rangle \\ &= \langle w, \langle x, z\rangle y\rangle \\ &= \langle w, T_{z,y}(x)\rangle \end{aligned}$$

Since this is true for any $w, x \in V$, we have $T^*_{y,z} = T_{z,y}$.

(d) Note that $T_{y,z}$ is self-adjoint if and only if $T_{y,z}^* = T_{y,z}$. From (c), we see that this is true if and only if $T_{y,z} = T_{z,y}$, which means

$$\langle x, y \rangle z = \langle x, z \rangle y$$

for any $x \in V$.

Suppose $y = cz$ for some $c \in \mathbb{R}$, then the above is trivial. Conversely, if we have $\langle x, y \rangle z = \langle x, z \rangle y$ for any $x \in V$. If $\langle x, z \rangle = 0$ for all $x \in V$, we have $z = 0$ and the statement is trivial, so we may take $y = 0$ and $c = 0$. If $\langle x, z \rangle \neq 0$ for some $x \in V$, then we have $y = \frac{\langle x,y \rangle}{\langle x,z \rangle} z$. Then we can take $c = \frac{\langle x,y \rangle}{\langle x,z \rangle}$ and we have $y = cz$. Moreover, we have

$$\langle x, z \rangle cz = \langle x, cz \rangle z = \langle x, z \rangle \bar{c}z,$$

and hence, $c = \bar{c}$, which means c is real. Hence, $T_{y,z}$ is self-adjoint if and only if $y = cz$ for some $c \in \mathbb{R}$.

4. Note that

$$\begin{aligned} \|x + ay\|^2 &= \langle x + ay, x + ay \rangle \\ &= \|x\|^2 + \bar{a} \langle x, y \rangle + a \langle y, x \rangle + |a| \|y\|^2 . \end{aligned}$$

Suppose x and y are orthogonal, we have

$$\|x + ay\|^2 = \|x\|^2 + |a| \|y\|^2 \geq \|x\|^2 .$$

Hence, $\|x\| \leq \|x + ay\|$.

Conversely, if $\|x\| \leq \|x + ay\|$, we have

$$\bar{a} \langle x, y \rangle + a \langle y, x \rangle + |a| \|y\|^2 = \|x + ay\|^2 - \|x\|^2 \geq 0$$

for all $a \in \mathbb{F}$. For $y = 0$, the statement is trivial. So let's assume $y \neq 0$. By taking $a = -\frac{\langle x,y \rangle}{\|y\|^2}$, we see that

$$\begin{aligned} -\frac{\overline{\langle x, y \rangle}}{\|y\|^2} \langle x, y \rangle - \frac{\langle x, y \rangle}{\|y\|^2} \langle y, x \rangle + \frac{| \langle x, y \rangle |^2}{\|y\|^4} \|y\|^2 &\geq 0 \\ -\frac{| \langle x, y \rangle |^2}{\|y\|^2} - \frac{| \langle x, y \rangle |^2}{\|y\|^2} + \frac{| \langle x, y \rangle |^2}{\|y\|^2} &\geq 0 \\ -\frac{| \langle x, y \rangle |^2}{\|y\|^2} &\geq 0 \\ | \langle x, y \rangle | &\leq 0, \end{aligned}$$

which means $\langle x, y \rangle = 0$. Hence, x and y are orthogonal.

5. (a) Suppose T is anti-self-adjoint. Then we have

$$T^*T = -T^2 = TT^*.$$

So, we see that T is normal. Moreover, if v is an eigenvector of T corresponding eigenvalue λ. Then we have

$$\lambda \langle v, v \rangle = \langle Tv, v \rangle = \langle v, T^*v \rangle = \langle v, -Tv \rangle = -\overline{\lambda} \langle v, v \rangle .$$

So, we see that λ is purely imaginary.

Conversely, if T is normal and all of its eigenvalues are purely imaginary. Then there exists an orthonormal basis β for V consisting of eigenvectors of T. Note that $[T]_\beta$ is a diagonal matrix with purely imaginary diagonal entries. So, we have

$$[T^*]_\beta = [T]_\beta^* = -[T]_\beta = [-T]_\beta.$$

Hence, $T^* = -T$ and T is anti-self-adjoint.

(b) Consider the characteristic polynomial of T and all its complex roots. Note that if α is a root, then $\overline{\alpha}$ is also a root. Since the number of roots is odd, there is some root satisfying $\alpha = \overline{\alpha}$. In other words, there is at least one real eigenvalue λ and $v \neq 0$ such that $Tv = \lambda v$. Now, T is anti-self-adjoint, we have $T^* = -T$ and

$$\lambda \langle v, v \rangle = \langle Tv, v \rangle = \langle v, T^*v \rangle = \langle v, -Tv \rangle = -\lambda \langle v, v \rangle ,$$

which means $\lambda = 0$. In other words, there is a nontrivial v satisfying $Tv = 0$. Hence, the dimension of the kernel of T is greater than 0.

6. To show that W is not a subspace of $\mathcal{L}(V)$, we find some elements from W such that their sum is outside W.

As $\dim(V) \geq 2$, we can find two orthonormal vectors, say v_1 and v_2. Consider the projection P_1 of vectors onto $\text{span}(\{v_1\})$ and the projection P_2 of vectors onto $\text{span}(\{v_1 + v_2\})$. Note that P_1 and P_2 are orthogonal projections, so they are self-adjoint. In particular, they are normal, so $P_1, P_2 \in W$. Consider P_1 and P_2 in W, we show that $P_1 + iP_2 \notin W$.

$$\begin{aligned}
&(P_1 + iP_2)^*(P_1 + iP_2) - (P_1 + iP_2)(P_1 + iP_2)^* \\
&= (P_1^* - iP_2^*)(P_1^* + iP_2^*) - (P_1^* + iP_2^*)(P_1^* - iP_2^*) \\
&= (P_1 - iP_2)(P_1 + iP_2) - (P_1 + iP_2)(P_1 - iP_2) \\
&= 2i(P_1P_2 - P_2P_1)
\end{aligned}$$

But $P_1P_2 - P_2P_1 \neq 0$ as

$$(P_1P_2 - P_2P_1)(v_2) = P_1P_2(v_2) = P_1(\frac{v_1 + v_2}{2}) = \frac{v_1}{2} \neq 0.$$

This shows that $P_1 + iP_2$ is not normal and $P_1 + iP_2 \notin W$. Hence, W is not a subspace of $\mathcal{L}(V)$.