Solution to Homework 4

Sec. 5.1

14. Note that the characteristic polynomial of A is $\det(A - \lambda I))$. By the fact that $\det(M^t) = \det(M)$, one can show that

$$\begin{aligned} \det(A - \lambda I) &= \det\left((A - \lambda I)^t\right) \\ &= \det(A^t - \lambda I) \end{aligned}$$

which means A and A^t have the same characteristic polynomial and hence they have the same eigenvalues.

Sec. 5.2

7. If we could diagonalize A, say $A = QAQ^{-1}$ where D is diagonal. Then we have $A^n = \left(QDQ^{-1}\right)^n = QD^nQ^{-1}$, where D^n can be easily expressed. So let's diagonalize A.

Consider the characteristic polynomial of A.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix} \\ &= (1-\lambda)(3-\lambda) - 2 \cdot 4 \\ &= \lambda^2 - 4\lambda - 5 \\ &= (\lambda + 1)(\lambda - 5) \end{aligned}$$

We see that $\lambda = -1$ and $\lambda = 5$ are two eigenvalues.

For $\lambda = -1$,

$$N(A + I) = N \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$$

For $\lambda = 5$,

$$N(A - 5I) = N \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

If we choose

$$\beta = \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

as the basis, then $[A]_\beta$ will be diagonal. Hence, let

$$Q = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}.$$

Then we have an expression for A^n.

$$A^n = QD^nQ^{-1} = Q \begin{pmatrix} (-1)^n & 0 \\ 0 & 5^n \end{pmatrix} Q^{-1}$$

8. Note that A is diagonalizable if we could find a basis consisting of eigenvectors of A.

   Now that $\dim(E_{\lambda_1}) = n - 1$, which means there exists a basis of eigenvectors correspond to eigenvalues λ_1. In other words, we have $n - 1$ linearly independent eigenvectors.

   Also, we have λ_2 to be an eigenvalue, that means there is some nonzero eigenvector, say v, corresponds to this value. Then $\beta \cup \{v\}$ are n linearly independent eigenvectors of A.

   So they form a basis consisting of eigenvectors of A. Hence, A is diagonalizable.

10. Note that the characteristic polynomial of T is

$$(\lambda_1 - t)^{m_1}(\lambda_2 - t)^{m_2} \cdots (\lambda_k - t)^{m_k}$$

   as $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the eigenvalues and $m_1, m_2, \ldots, m_k$ are the corresponding multiplicities.

   Consider upper triangular matrix $[T]_\beta$, the characteristic polynomial of $[T]_\beta$ is just $\det([T]_\beta - tI)$. Note that $[T]_\beta - tI$ is also an upper triangular matrix. So we have

$$\det([T]_\beta - tI) = (d_1 - t)(d_2 - t) \cdots (d_n - t),$$

   where $n = \dim(V)$ and $d_1, d_2, \ldots, d_n$ are the diagonal entries of $[T]_\beta$.

   However, we know that the characteristic polynomial of T does not depends on the choice of basis. So we must have

$$(\lambda_1 - t)^{m_1}(\lambda_2 - t)^{m_2} \cdots (\lambda_k - t)^{m_k} = (d_1 - t)(d_2 - t) \cdots (d_n - t),$$

   which means each d_j corresponds to one of the λ_is. Moreover, there are exactly m_i of d_js appear to be λ_i.

11. (a) By similar arguments as in the above exercise, we know that the diagonal entries of A are $\lambda_1, \lambda_2, \ldots, \lambda_k$ and each λ_i occurs m_i times. So we have

$$\operatorname{tr}(A) = \sum_{i=1}^{k} m_i \lambda_i.$$

(b) Since the determinant of an upper triangular matrix is just the product of the diagonal entries. So we have

$$\det(A) = (\lambda_1)^{m_1}(\lambda_2)^{m_2} \cdots (\lambda_k)^{m_k}.$$

13. (a) Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. Note that

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector of A and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of A^t, both corresponds to the same eigenvalue 0. However, the eigenspace of A and A^t are not the same as

$$E_0 = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} \text{ and } E_0 = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

are the eigenspaces for A and A^t respectively.

17. (a) As T and U are simultaneously diagonalizable, they share a basis such that $[T]_\gamma$ and $[U]_\gamma$ are diagonal matrices.
By changing the basis, we see that

$$[T]_\gamma = [I]_\beta^\gamma [T]_\beta [I]_\gamma^\beta,$$

where $[I]_\gamma^\beta$ is invertible and $\left([I]_\gamma^\beta\right)^{-1} = [I]_\beta^\gamma$. So if we take $Q = [I]_\gamma^\beta$, then $Q^{-1}[T]_\beta Q = [T]_\gamma$ is a diagonal matrix.
As the transition matrices are the same, we have $Q^{-1}[U]_\beta Q = [U]_\gamma$ too. Hence, there exists an invertible matrix Q such that both $Q^{-1}[T]_\beta Q$ and $Q^{-1}[U]_\beta Q$ are diagonal.
Since β is arbitrary, $[T]_\beta$ and $[U]_\beta$ are simultaneously diagonalizable for any ordered basis β.

(b) If A and B are simultaneously diagonalizable, then there exists an invertible matrix Q such that $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal.
Let β be the standard basis of $\mathbb{F}^n$ and γ be the columns of Q. Then γ is a basis as Q is invertible. Note that $[L_A]_\beta$ and $[I]_\gamma^\beta$ are just A and Q respectively, so we have

$$[L_A]_\gamma = [I]_\beta^\gamma [L_A]_\beta [I]_\gamma^\beta = Q^{-1}AQ,$$

which is a diagonal matrix.

Similarly, we have

$$[L_B]_\gamma = [I]_\beta^\gamma [L_B]_\beta [I]_\gamma^\beta = Q^{-1}BQ.$$

Hence, L_A and L_B are simultaneously diagonalizable.

18. (a) If T and U are simultaneously diagonalizable, then there exists a basis $\beta = \{v_1, v_2, \ldots, v_n\}$ such that $[T]_\beta$ and $[U]_\beta$ are diagonal matrices. In other words, $Tv_i = \lambda_i v_i$ and $Uv_i = \sigma v_i$ for $i = 1, 2, \ldots, n$. Observe that

$$TUv_i = \sigma_i \lambda_i v_i = \lambda_i \sigma_i v_i = UTv_i$$

for $i = 1, 2, \ldots, n$. Since β is a basis, for every $x \in V$, we can express x as a linear combination of v_i.

$$x = a_1 v_1 + a_2 v_2 + \cdots + a_n v_k$$

Then it is easy to check that $TUx = UTx$.

$$\begin{aligned} TUx &= TU(a_1 v_1 + a_2 v_2 + \cdots + a_n v_k) \\ &= a_1 TUv_1 + a_2 TUv_2 + \cdots + a_n TUv_k \\ &= a_1 UTv_1 + a_2 UTv_2 + \cdots + a_n UTv_k \\ &= UTx \end{aligned}$$

Since x is arbitrary, T and U commute.

(b) If A and B are simultaneously diagonalizable, then there exists an invertible matrix Q such that $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal. Note that diagonal matrices commute.

$$\left(Q^{-1}AQ\right)\left(Q^{-1}BQ\right) = \left(Q^{-1}BQ\right)\left(Q^{-1}AQ\right)$$

As Q is invertible, we have $AB = BA$, which means A and B commute.