Solution to Homework 3

5.1

3. (c) i. We want to solve $\det(A - \lambda I) = 0$ for some λ

$$\det \begin{pmatrix} i-\lambda & 1 \\ 2 & -i-\lambda \end{pmatrix} = 0.$$

Easily, one can get $\lambda^2 - 1 = 0$. So eigenvalues are -1 and 1.

ii. For $\lambda = -1$, we want to solve $Av = -v$ for some nonzero $v \in \mathbb{C}^2$.

$$Av = -v \Rightarrow (A+I)v = 0 \Rightarrow \begin{pmatrix} 1+i & 1 \\ 2 & 1-i \end{pmatrix} v = 0$$

One possible choice of v is $\begin{pmatrix} 1 \\ -1-i \end{pmatrix}$. So $\begin{pmatrix} 1 \\ -1-i \end{pmatrix}$ is an eigenvector with respect to eigenvalue $\lambda = -1$.
Similarly, for $\lambda = 1$, we want to find a nonzero eigenvector.

$$Av = v \Rightarrow \begin{pmatrix} -1+i & 1 \\ 2 & -1-i \end{pmatrix} v = 0$$

One can choose v to be $\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$. So $\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$ is an eigenvector with respect to eigenvalue $\lambda = 1$.
Obviously, $\beta = \left\{ \begin{pmatrix} 1 \\ -1-i \end{pmatrix}, \begin{pmatrix} 1 \\ 1-i \end{pmatrix} \right\}$ is a basis for $\mathbb{C}^2$ consisting of eigenvectors of A.

iii. Let γ be the standard basis for $\mathbb{C}^2$. If we take $Q = [I]_\beta^\gamma$, which is invertible, then $Q^{-1}AQ = [I]_\gamma^\beta [L_A]_\gamma [I]_\beta^\gamma = [L_A]_\beta$ will be a diagonal matrix as β are the eigenvectors of A.
Hence, we have

$$Q = \begin{pmatrix} 1 & 1 \\ -1-i & 1-i \end{pmatrix}$$

and

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(d) i. Again we solve for $\det(A - \lambda I) = 0$.

$$\det \begin{pmatrix} 2-\lambda & 0 & -1 \\ 4 & 1-\lambda & -4 \\ 2 & 0 & -1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)\det \begin{pmatrix} 2-\lambda & -1 \\ 2 & -1-\lambda \end{pmatrix} = 0$$

So we have $(1-\lambda)\left((2-\lambda)(-1-\lambda)+2\right) = 0 \Rightarrow \lambda(\lambda-1)^2 = 0$. So 0 and 1 are the eigenvalues of A.

ii. For $\lambda = 0$, we solve for some nonzero v as eigenvector.

$$\begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} v = 0$$

We choose $v = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$.

For $\lambda = 1$, we solve for some nonzero v.

$$\begin{pmatrix} 1 & 0 & -1 \\ 4 & 0 & -4 \\ 2 & 0 & -2 \end{pmatrix} v = 0$$

Then $v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are two nonzero solutions and linearly independent to each other.

Hence, $\beta = \left\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

iii. Let γ be the standard basis for $\mathbb{R}^3$. Similarly, we set

$$Q = [I]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 4 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

Then

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

will be a diagonal matrix consisting of eigenvalues of A in corresponding order.

4. (e) Let $\gamma = \{1, x, x^2\}$ be the standard basis for $P_2(\mathbb{R})$. Then

$$[T]_\gamma = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix}.$$

Next, we solve $\det([T]_\gamma - \lambda I) = 0$ for some λ.

$$\begin{pmatrix} 1-\lambda & 3 & 9 \\ 1 & 3-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

Easily, one can get

$$0 = (2-\lambda)((1-\lambda)(3-\lambda) - 3) = \lambda(2-\lambda)(\lambda - 4).$$

Then for each λ, we look for nonzero vectors v such that $[T]_\gamma v = \lambda v$.
For $\lambda = 0$, we have

$$\begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix} v = 0.$$

One possible solution is $v = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$. So we see that $-3 + x$ is an eigenvector of T.
For $\lambda = 2$, we have

$$\begin{pmatrix} -1 & 3 & 9 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} v = 0.$$

One possible solution is $v = \begin{pmatrix} -3 \\ -13 \\ 4 \end{pmatrix}$. So we see that $-3 - 13x + 4x^2$ is another eigenvector of T.
For $\lambda = 4$, we have

$$\begin{pmatrix} -3 & 3 & 9 \\ 1 & -1 & 4 \\ 0 & 0 & -2 \end{pmatrix} v = 0.$$

One possible solution is $v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. So we see that $1 + x$ is another eigenvector of T.
Hence $\beta = \left\{-3 + x, -3 - 13x + 4x^2, 1 + x\right\}$ is a basis for $P_2(\mathbb{R})$ consisting of eigenvectors of T. So $[T]_\beta$ is a diagonal matrix.

$$[T]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

(h) Note that we are solving $Tv = v$ for some λ and $\mathbf{0} \neq v \in M_{2\times 2}(\mathbb{R})$.

$$\begin{pmatrix} d & b \\ c & a \end{pmatrix} = T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let $\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be the standard basis for $M_{2\times 2}(\mathbb{R})$. Then one can easily write down

$$[T]_\gamma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Next, by solving $\det([T]_\gamma - \lambda I) = 0$,

$$\det \begin{pmatrix} -\lambda & 0 & 0 & 1 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{pmatrix} = 0,$$

one could get λ to be -1 or 1.
For $\lambda = -1$, we have

$$\begin{pmatrix} d & b \\ c & a \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}.$$

So one possible solution is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
For $\lambda = 1$, we have

$$\begin{pmatrix} d & b \\ c & a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

So $\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ are possible linearly independent solutions.
Together, we have a basis of eigenvectors of T for $M_{2\times 2}(\mathbb{R})$.

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Moreover, we have

$$[T]_\beta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

8. (a) Note that zero is an eigenvalue of T if and only if there exists some nonzero vector v such that $Tv = 0v = 0$. This is equivalent to say that there exists some nonzero vector

$$v \in N(T - \lambda I) = N(T),$$

that is $N(T) \neq \{\mathbf{0}\}$. So T is not invertible. In other words, T is invertible if and only if zero is not an eigenvalue of T.

(b) Again λ is an eigenvalue if and only if $T(v) = \lambda v$ for some nonzero vector v. As T is invertible, from the above, we see $\lambda \neq 0$. So this means

$$\lambda v = \lambda^{-1}T^{-1}(T(v)) = T^{-1}v,$$

which means that λ^{-1} is an eigenvalue of T^{-1}.

(c) i. First, we show that M is invertible if and only if $\lambda = 0$ is not an eigenvalue. This is true because λ is an eigenvalue if and only if there exists some nonzero vector v such that

$$Mv = \lambda v.$$

But λ is just zero, this means there is some nontrivial solution to the system

$$Mv = 0,$$

that is M is not invertible. In order words, M is invertible if and only if zero is not an eigenvalue of M.

ii. Second, we prove that λ^{-1} is an eigenvalue of M^{-1}. As M is invertible, we have $\lambda \neq 0$ by the above argument. Since there is some nonzero vector v such that $Mv = \lambda v$, hence we can multiply both sides by M^{-1} and λ^{-1}.

$$\lambda^{-1}v = \lambda^{-1}M^{-1}Mv = M^{-1}v$$

This is equivalent to say that λ^{-1} is an eigenvalue of M^{-1}.

12. (a) Suppose A is similar to B, there exists some invertible matrix P such that

$$A = P^{-1}BP$$

with $\det(P) \neq 0$. Then

$$\begin{aligned}\det(A - \lambda I) &= \det(P^{-1}BP - \lambda I)\\ &= \det(P^{-1}BP - P^{-1}(\lambda I)P)\\ &= \det(P^{-1}(B - \lambda I)P)\\ &= \det(P)^{-1}\det(B - \lambda I)\det(P)\\ &= \det(B - \lambda I)\end{aligned}$$

(b) Note the representations of a linear operator T are similar matrices. In order words $[T]_\alpha$ is similar to $[T]_\beta$ for any choices of bases α and β. (This is true as for bases α and β, $[I]_\alpha^\beta$ is invertible and $[I]_\beta^\alpha = \left([I]_\alpha^\beta\right)^{-1}$.)
Then, by the above part, we see that the characteristic polynomial is well-defined.

Hence, the characteristic polynomial is independent of the choice of basis for V.

18. (a) Note that if B is invertible, we can "factor" B out from $A + cB$. Then, by considering $\det(A + cB)$, we have

$$\det(A + cB) = \det(B)\det(B^{-1}A + xI),$$

which is a polynomial of x over $\mathbb{C}$.

By the fundamental theorem of algebra, there must be a root of the polynomial, say c, such that $\det(B^{-1}A + cI) = 0$.

Hence, there exists some scalar $c \in \mathbb{C}$ such that $\det(A + cB) = 0$, in other words $A + cB$ is not invertible.

(b) From the above, we see that if B is invertible, then $A + cB$ will not be invertible for some $c \in \mathbb{C}$. So we choose B to be some nonzero matrix which is not invertible.

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Let $A = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$. Then

$$A = \begin{pmatrix} i & j \\ k & l \end{pmatrix} \text{ and } A + cB = \begin{pmatrix} i + c & j + c \\ k & l \end{pmatrix}$$

are invertible for all $c \in \mathbb{C}$. In other words, we need $il \neq jk$ and $(i + c)l \neq (j + c)k$. One possible choice is to choose $k = l \neq 0$, then any $i \neq j$ would give a feasible solution. So we may choose

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and A and $A + cB$ would be invertible for all $c \in \mathbb{C}$.

20. By definition, we have $\det(A - tI) = f(t)$. So when $t = 0$, we get $\det(A) = f(0) = a_0$. In other words, A is invertible if and only if $a_0 \neq 0$.

21. (a) Let's prove main statement by induction.
For $n = 2$, we have $f(t) = \det(A - tI) = (A_{11} - t)(A_{22} - t) - A_{12}A_{21}$. As $A_{12}A_{21}$, a constant, is a polynomial of 0 degree, the statement is true for $n = 2$.

Assume the statement is true for $n = k - 1$. We prove the statement for $n = k$. First, expand the determinant along the first row.

$$\det(A - tI) = (A_{11} - t)\det(\tilde{A}_{11} - t\tilde{I}) + \sum_{j=2}^{n}(-1)^{1+j}A_{1j}\det(B_{1j})$$

(Here $\tilde{I}$ is just $I_{(n-1)\times(n-1)}$.)

$$\begin{pmatrix} A_{11} - t & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} - t & A_{23} & \cdots & A_{2n} \\ A_{31} & A_{32} & A_{33} - t & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & A_{n3} & \cdots & A_{nn} - t \end{pmatrix}$$

We observe that the first columns of each B_{1j} is independent of t. So we can expand the determinant along the first row.

$$\det(B_{1j}) = \sum_{k=1}^{n-1}(B_{1j})_{ik}\det(\widetilde{(B_{1j})}_{ik})$$

Note that $\widetilde{(B_{1j})}_{ik}$ is an $(n-2)\times(n-2)$ matrix with at most one entry involving t, $\det(\widetilde{(B_{1j})}_{ik})$ is a polynomial in t of degree not greater than $n-2$ †, so is $\det(B_{1j})$. So the second part of $\det(A-tI)$ is a polynomial of degree at most $n-2$.

The first part follows easily from the induction hypothesis. We have

$$\det(\tilde{A}_{11} - t\tilde{I}) = (A_{22} - t)\cdots(A_{nn} - t) + q(t),$$

where $q(t)$ is a polynomial of degree at most $n-2$.

Hence, we get

$$\begin{aligned}\det(A - tI) =& (A_{11} - t)(A_{22} - t)\cdots(A_{nn} - t) \\ & \underbrace{+\, q(t) + \sum_{j=2}^{n}(-1)^{1+j}A_{1j}\det(B_{1j}),}_{p(t)}\end{aligned}$$

where $p(t)$ is a polynomial of degree at most $n-2$. The statement then follows by induction.

† We claim that if $B \in M_{n\times n}(\mathbb{R})$ is a matrix such that in each row, at most one entry involves variable t, then $\det(B)$ is a polynomial in t of degree not greater than n.

When $n = 1$, this is obviously true.

Suppose the claim holds for $n = k - 1$. When $n = k$, we note that, by expanding the determinant along some row,

$$\det(B) = \sum_{j=1}^{n} B_{ij} \det(\tilde{B}_{ij})$$

for some i. It is easy to see that $\tilde{B}_{ij}$ is a matrix with at most one entry involving t in each row. So, by induction hypothesis, $\det(\tilde{B}_{ij})$ is a polynomial in t with degree not greater than $n - 1$.

As a result $\det(B)$ is a polynomial in t with degree not greater than n. Hence, by induction, the claim is true.

(b) From the Exercise 20, we have

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

From the above part, we have

$$\begin{aligned} f(t) =& (A_{11} - t)(A_{22} - t) \cdots (A_{nn} - t) + q(t) \\ =& (-1)^n t^n + (A_{11} + A_{22} + \cdots + A_{nn})(-1)^{n-1}t^{n-1} + r(t), \end{aligned}$$

where $r(t)$ are terms of degree at most $n - 2$.

Hence, we see that

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii} = (-1)^{n-1}a_{n-1}.$$

5.2

2. (e) First, we look at the characteristic polynomial of A.

$$\det \begin{pmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & -1 \\ 0 & 1 & 1-\lambda \end{pmatrix} = 0$$

We then get

$$-\lambda^3 + \lambda^2 - \lambda + 1 = (1 - \lambda)(\lambda^2 + 1) = 0,$$

which does not split in $\mathbb{R}$. So we conclude that A is not diagonalizable.

3. (c) Let γ be the standard basis for $\mathbb{R}^2$. Then one can easily write down

$$[T]_\gamma = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then we see that the characteristic polynomial of T, which is the same as that of $[T]_\gamma$, does not split over $\mathbb{R}$.

$$\det\begin{pmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda)(\lambda^2+1) = 0$$

Hence, T is not diagonalizable over $\mathbb{R}$.

12. (a) Let E_λ denotes the eigenspace of T corresponding to λ and $F_{\lambda^{-1}}$ denotes the eigenspace of T^{-1} corresponding to λ^{-1}.

    Recall that for any eigenvalue λ of T, λ^{-1} is an eigenvalue of T^{-1}. So for any $v \in E_\lambda$, it is an eigenvector of T. Then it is an eigenvector of T^{-1}, which means $v \in F_{\lambda^{-1}}$.

    Similarly, one can show $v \in F_{\lambda^{-1}}$ implies $v \in E_\lambda$.

    Hence, $E_\lambda = F_{\lambda^{-1}}$.

    (b) If T is diagonalizable, then there exists a basis β for V consisting of eigenvectors of T. As T is invertible, β is also a basis for V consisting of eigenvectors of T^{-1}.

    Hence, T^{-1} is also diagonalizable.