Lecture 24 : Jordan Canonical Form (Part 2)

Last time: • JCF : $\begin{pmatrix} A_1 & & & O \\ & A_2 & & \\ & & \ddots & \\ O & & & A_k \end{pmatrix}$ and

each block A_i is given by:

$$A_i = \begin{pmatrix} \lambda_i & 1 & & O \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ O & & & \lambda_i \end{pmatrix}$$

• To compute JCF, need to consider:

$$K_\lambda = \{ \vec{x} \in V : (T - \lambda I)^p(\vec{x}) = \vec{0} \text{ for some positive integer } p \in \mathbb{N} \}$$

eigenvalue

• $\dim(K_\lambda) = \underbrace{\text{multiplicity}}_{m}$ and $K_\lambda = N((T-\lambda I)^m)$

• Finding JC basis $\longleftrightarrow$ finding basis = disjoint union of cycles.

Example 1: Let $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 3 \end{pmatrix}$

Char poly of $A = (2-t)^4$. $\therefore \lambda = 2$ is the ONLY eigenvalue.

$\therefore \dim(K_\lambda) = 4$.

Need to find basis β of K_λ = disjoint union of cycles.

Few possibilities: ① One cycle of length 4.

② Two cycles of length $\left\langle \begin{array}{l} 2+2 \\ 1+3 \end{array} \right.$ ($\Rightarrow$ 2 lin. ind. eigenvectors)

③ Three cycles of length 1, 1, 2 ($\Rightarrow$ 3 lin. ind. eigenvectors)

④ Four cycles of length 1, 1, 1, 1 ($\Rightarrow$ 4 lin. ind. eigenvectors)

We can check that $\dim(E_\lambda) = 4 - \underbrace{\operatorname{rank}(A-2I)}_{\substack{= \#\text{ of lin. ind. cols} \\ = 3}} = 1.$

$\therefore$ ②, ③, ④ are impossible.

$\therefore \beta = \{(T-2I)^3\vec{v}, \dots, \vec{v}\}.$

Need to find $\vec{v} \neq 0 \in K_\lambda = N((A-2I)^4)$ such that $(A-2I)^i(\vec{v}) \neq \vec{0}$ for $i = 1, 2, 3$.

We can check: $\vec{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ satisfies this condition.

Hence, $\beta = \left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{=(T-2I)^3\vec{v}}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{=\vec{v}} \right\}$ and $[T]_\beta = \begin{pmatrix} 2 & 1 & & 0 \\ & 2 & 1 & \\ 0 & & 2 & 1 \\ & & & 2 \end{pmatrix}$

Better way to find lengths of cycles: Dot diagram

Main step to find JCF: Find basis β_i of K_{λ_i} = disjoint union of cycles.

Let $\beta_i = \gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_{n_i}$. Assume: $|\gamma_1| \geq |\gamma_2| \geq \dots \geq |\gamma_{n_i}|$

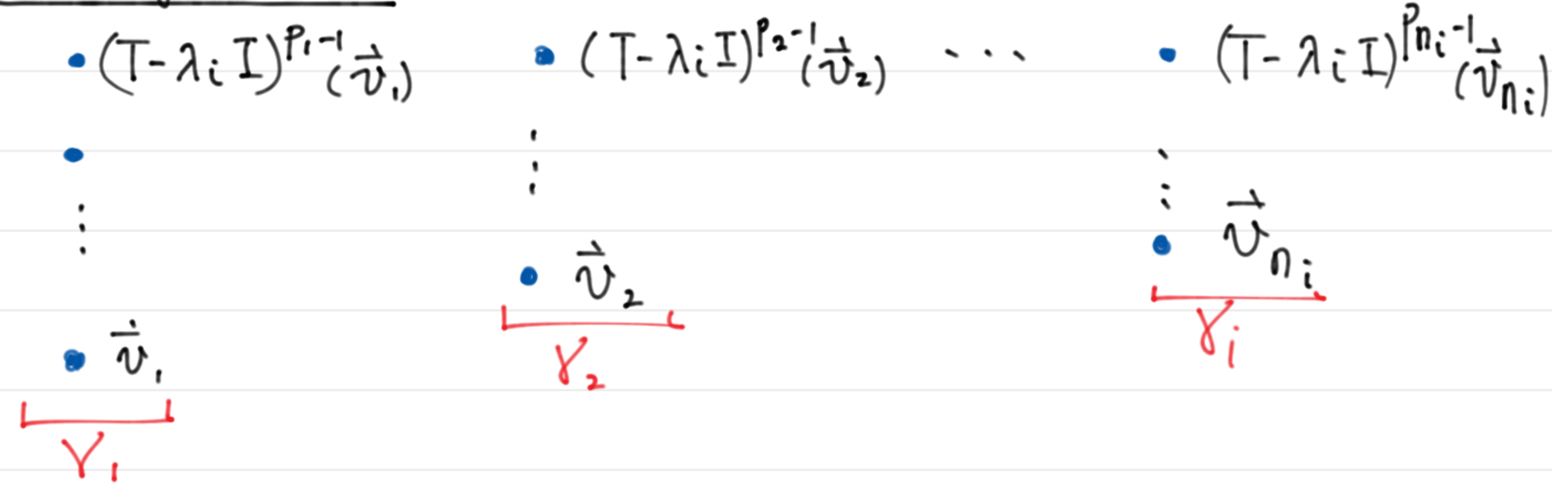
Let $T_i = T|_{K_{\lambda_i}}$.

Let $\gamma_1 = \{(T-\lambda_i I)^{p_1-1}(\vec{v}_1), \dots, \vec{v}_1\}$

$\vdots$

$\gamma_{n_i} = \{(T-\lambda_i I)^{p_{n_i}-1}(\vec{v}_{n_i}), \dots, \vec{v}_{n_i}\}$

Dot diagram of T_i :



<u>Properties of dot diagram</u>: Let r_j = # of dots in the j-th row.

① $r_1 = \dim(E_{\lambda_i}) = \dim(V) - \operatorname{rank}((T-\lambda_i I))$

② $r_j = \operatorname{rank}((T-\lambda_i I)^{j-1}) - \operatorname{rank}((T-\lambda_i I)^{j})$

<u>Remark</u>: • For any eigenvalue λ_i, the dot diagram of T_i is unique (up to the ordering of cycles)

• $\therefore$ JCF is unique.

<u>Example 2</u>: Let's go back to $A = \begin{pmatrix} 0 & -1 & 0 \\ 4 & 4 & 0 \\ -2 & -3 & 6 \end{pmatrix} \in M_{3\times3}(\mathbb{R})$. Find JCF.

<u>Solution</u>: <u>Step 1</u>: Compute char poly : $g(t) = (6-t)(2-t)^2$

$\therefore \lambda_1 = 6,\ \lambda_2 = 2$

<u>Step 2</u>: Draw dot diagrams for K_{λ_i}. Let $T = L_A$.

$\dim(K_{\lambda_1}) = 1$. $\therefore$ Dot diagram for $T|_{K_{\lambda_1}}$ is : $\bullet$

$\dim(K_{\lambda_2}) = 2$. Now, $r_1 = 3 - \operatorname{rank}(A-2I) = 3 - \underbrace{\operatorname{rank}\begin{pmatrix} -2 & -1 & 0 \\ 4 & 2 & 0 \\ -2 & 3 & 4 \end{pmatrix}}_{=2} = 1$

$\therefore$ Dot diagram for $T|_{K_{\lambda_2}}$ is : $\begin{matrix}\bullet \\ \bullet\end{matrix}$

Step 3: Find β_1 and β_2.

β_1: Find $\vec{v} \neq \vec{0} \in K_{\lambda_1} = E_{\lambda_1} = N(A-6I)$. Pick $\vec{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

β_2: Find $\vec{v} \neq \vec{0}$ such that $\vec{v} \in N((A-2I)^2)$ but $\vec{v} \notin N(A-2I)$

We can check that: $A-2I = \begin{pmatrix} -2 & -1 & 0 \\ 4 & 2 & 0 \\ -2 & 3 & 4 \end{pmatrix}$ and $(A-2I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -16 & -16 & 16 \end{pmatrix}$

$\therefore N((A-2I)^2) = \operatorname{span}\{\underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}_{\text{basis}}\}$. Take $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \notin N(A-2I)$

$\therefore \beta_2 = \left\{(A-2I)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\} = \left\{\begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$.

Step 4: Define JC basis and JCF.

$\therefore \beta = \beta_1 \cup \beta_2 = \left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$ is a JC basis.

$$JCF = \begin{pmatrix} \boxed{6} & & \\ & 2 & 1 \\ & & 2 \end{pmatrix} = Q^{-1}AQ \; ; \; Q = \begin{pmatrix} 0 & -2 & 1 \\ 0 & 4 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

<u>Example 3</u>: $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 3 \end{pmatrix}$. Let $T = L_A$.

Step 1: Char poly: $(2-t)^4 \quad \therefore \lambda_1 = 2$.

Step 2: Draw dot diagram for $T|_{K_{\lambda_1}}$:

$r_1 = 4 - \operatorname{rank}(A-2I) = 4 - \underset{= 3}{\underbrace{\operatorname{rank}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & -1 & 0 \end{pmatrix}}} = 4 - 3 = 1$

$\therefore$ Dot diagram for β_1 is: $\begin{matrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{matrix}$

Step 3: Find β_1: Find $\vec{v} \neq \vec{0}$ such that $(A-2I)^4\vec{v} = \vec{0}$ but $(A-2I)^i(\vec{v}) \neq \vec{0}$ for $i = 1, 2, 3$.

Now, $(A-2I)^4 = O$ (Check) $\therefore N((A-2I)^4) = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$.

Pick $\vec{v} = \vec{e}_1$. We can check: $(A-2I)^i(\vec{v}) \neq \vec{0}$ for $i = 1, 2, 3$.

$\therefore \beta_1 = \{(A-2I)^3(\vec{e}_1), (A-2I)^2(\vec{e}_1), (A-2I)\vec{e}_1, \vec{e}_1\}$

$$= \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Step 4: Define JC basis and JCF.

Let $\beta = \beta_1$.

Then: $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} = Q^{-1}AQ$; $Q = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

<u>Exercise</u>: Find JCF of $A = \begin{pmatrix} 6 & -1 & 1 & 1 & 0 \\ 2 & 7 & 1 & 1 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 1 & -1 & 5 & 0 \\ 0 & 1 & -1 & -1 & 6 \end{pmatrix}$