Lecture 2 : Revision (2)

System of linear equations : (unknowns: $x_1, \dots, x_n$)

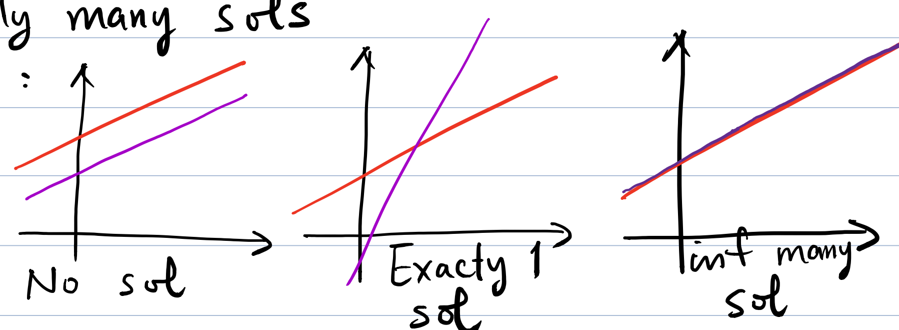
$$(*)\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

There are 3 possibilities :

(1) No solution (2) Exactly 1 solution

(3) Infinitely many sols

Geometrically :



(*) can be written as:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \quad \text{or} \quad \boxed{A\vec{x} = \vec{b}}$$

$m \times n$ matrix

- Operations of matrices :
  - $\underset{m\times n}{A \pm B}$ gives $m \times n$ matrix
  - aA gives $m \times n$ matrix
  - $\underset{m\times n}{A} \times \underset{n\times k}{B}$ gives $m \times k$ matrix

But $AB \neq BA$ in general (Non-commutativity)

- A is called <u>invertible</u> if $\exists A^{-1}$ such that:

$$AA^{-1} = \underset{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}{I} = A^{-1}A$$

<u>Study of matrices</u>:

- Elementary operations: Type I : ⓘ $\leftrightarrow$ ⓙ
  Type II : $\alpha \times$ ⓘ
  Type III : $\alpha \times$ ⓘ $+$ ⓙ
- Elementary matrices : obtained from performing an elementary operation of Type I, II or III on I.
- Elementary row (col) operations $\Leftrightarrow$ Left (right) matrix multiplication by elementary matrix.
- By elementary row operations, every matrix has its unique RREF (reduced row echelon form):

RREF $R = \begin{pmatrix} 1 & * & 0 & * & 0 & * & * & 0 & * & * \\ 0 & \cdots & 1 & * & 0 & * & * & \vdots & & * \\ 0 & \cdots & 0 & \cdots & 1 & * & * & \vdots & & \\ & & & & 0 & & & 1 & & \\ & & & & \vdots & & & 0 & & \\ 0 & \cdots & & & 0 & & & 0 & & 0 \end{pmatrix}$

Assume R is the RREF of A.

- Then $\operatorname{rank}(A) = \#$ of cols with leading 1's.
- Let $j_1, j_2, \ldots, j_r$ = cols of R with leading 1's.
  Then: $\{\vec{a}_{j_1}, \vec{a}_{j_2}, \ldots, \vec{a}_{j_r}\}$ form a basis

of the col space of A ($\vec{a}_{j_k} = j_k$ col of A)

- If col k of R is: $d_1\vec{e}_1 + d_2\vec{e}_2 + \dots + d_r\vec{e}_r$, then col k of A is: $d_1\vec{a}_{j_1} + d_2\vec{a}_{j_2} + \dots + d_r\vec{a}_{j_r}$ (Refer to P. 191)
- Every invertible matrix A is a product of elementary matrices: $A = E_1 E_2 \dots E_n$ (P. 161)
- Let P and Q be invertible. Then:

$$\text{Rank}(PAQ) = \text{Rank}(A) \quad \text{(P. 153)}$$

- A can be further analyzed by determinant:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(\tilde{A}_{ij})$$

(i,j)-entry of A

Removing the i^{th} row, j^{th} col of A.

(Recursive formula to compute det(A))

For 2x2: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

For 3x3: $\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$

For nxn: recursively defined!

Thm: $\det(AB) = \det(A)\det(B)$

$\det(A^T) = \det(A)$

Thm: A is invertible iff $\det(A) \neq 0$

- Inverse of A can be computed:

$$(A \mid I) \xrightarrow[\text{operation}]{\text{Row}} \underbrace{(I \mid A^{-1})}_{\text{RREF of } (A|I).}$$