Lecture 19(1): Self-adjoint operators

Definition: Let T be linear operator on an inner product space V. Then: T is called self-adjoint (hermitian) iff $T = T^*$.

A matrix $A \in M_{n\times n}(\mathbb{F})$ is called self-adjoint (hermitian) iff $A = A^*$.

Goal: Self-adjoint $\longrightarrow$ Real inner product space has orthonormal basis of eigenvectors.

Lemma: Let T be a self-adjoint operator on a fin-dim inner product space (over $\mathbb{F} = \mathbb{C}$). Then:

(a) Every eigenvalue of T is real

(b) Suppose V is a real inner product space. Then the characteristic polynomial of T splits. (over $\mathbb{F} = \mathbb{R}$)

Proof: (a) Let $T(\vec{x}) = \lambda \vec{x}$ with $\vec{x} \neq \vec{0}$

(eigenvector exists for complex inner product space)

Since T is self-adjoint, T is also normal.

Thus, $\lambda \vec{x} = T(\vec{x}) \overset{\text{(self-adjoint)}}{=} T^*(\vec{x}) \overset{\text{(normal)}}{=} \bar{\lambda}\vec{x}$.

This implies $\lambda = \bar{\lambda}$ and so λ is real.

(b) Let $n = \dim(V)$ and $\beta =$ orthonormal basis for V.

Let $A := [T]_\beta$.

$\therefore$ A is self-adjoint since T is self-adjoint.

Define $T_A : \mathbb{C}^n \to \mathbb{C}^n$ by $T_A(\vec{x}) = A\vec{x}$ where $\vec{x} \in \mathbb{C}^n$.

Let $\gamma =$ standard ordered basis for $\mathbb{C}^n$.

Then: $[T_A]_\gamma = A$, which is self-adjoint.

By (a), eigenvalue of T_A is real.

Also, by fundamental Thm of Alg, the char poly of T_A splits over $\mathbb{F} = \mathbb{C}$.

Since all eigenvalues of T_A are real, the char poly of T splits over $\mathbb{F} = \mathbb{R}$.

<u>Main Theorem</u>

<u>Theorem:</u> Let T be a lin operator on a fin-dim real inner product space. T is self-adjoint if and only if $\exists$ orthonormal basis of eigenvectors of T.

<u>Proof:</u> Suppose T is self-adjoint. By lemma, the char poly splits. By Schur's lemma, $\exists$ orthonormal basis β for V such that $A := [T]_\beta$ is upper triangular.

Now, $A^* = ([T]_\beta)^* = [T^*]_\beta = [T]_\beta = A$

Thus, both A and A^* are upper triangular.

We conclude that A is diagonal. This implies all elements in β must be eigenvectors.

The converse is trivial. Suppose $\exists$ orthonormal basis β of eigenvectors. Then, $[T]_\beta$ is real diagonal.

Hence, $[T^*]_\beta = ([T]_\beta)^* = ([T]_\beta)^T = [T]_\beta$

$\therefore \quad T^* = T.$