Lecture 11: More about invariant subspaces

Example 1: Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(a,b,c) = (2b+3c,\ 3a+3c,\ 8c)$

Let $W = \{(s,t,0) : s,t \in \mathbb{R}\}$ ← T-invariant. Let $\gamma = \{\vec{e}_1, \vec{e}_2\}$, $\beta = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

Then: $[T_W]_\gamma = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} \rightsquigarrow$ Char poly of $T_W = t^2 - 6 = g(t)$

Now, $[T]_\beta = \begin{pmatrix} \boxed{\begin{matrix} 0 & 2 \\ 3 & 0 \end{matrix}} & \begin{matrix} 3 \\ 3 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 8 \end{pmatrix}$ ($[T_W]_\gamma$ =) $\rightsquigarrow$ Char poly of $T = (t^2-6)(8-t) = f(t)$

$\therefore\ g(t) \mid f(t)$.

Theorem 1: Let $T: V \to V$ (finite-dim). Let W = T-cyclic subspace generated by $\vec{v} \neq 0 \in V$. Let $k = \dim(W)$. Then:

1. $\{\vec{v}, T(\vec{v}), \ldots, T^{k-1}(\vec{v})\}$ is a basis for W.
2. Let $a_0\vec{v} + a_1 T(\vec{v}) + \ldots + a_{k-1} T^{k-1}(\vec{v}) + T^k(\vec{v}) = \vec{0}$.

   Then, the char poly of T_W is: $(-1)^k(a_0 + a_1 t + \ldots + a_{k-1}t^{k-1} + t^k)$

Proof: $\vec{v} \neq \vec{0}$. So, $\{\vec{v}\}$ is lin. ind. Let j = largest integer such that $\beta = \{\vec{v}, T(\vec{v}), \ldots, T^{j-1}(\vec{v})\}$ is lin. ind.
(Must exist since V is finite-dim)

Let $X = \text{span}(\beta)$. Then, β = basis of X. Also, $T^j(\vec{v}) \in X$ since $T^j(\vec{v})$ can be written as lin. comb. of β.

We show that X is T-invariant:

Let $\vec{w} \in X$. Then: $\vec{w} = a_0\vec{v} + a_1 T(\vec{v}) + \ldots + a_{j-1}T^{j-1}(\vec{v})$

$\Rightarrow T(\vec{w}) = a_0 \underbrace{T(\vec{v})}_{\in X} + a_1 \underbrace{T^2(\vec{v})}_{\in X} + \ldots + a_{j-1} \underbrace{T^j(\vec{v})}_{\in X} \in X$

Now, W = T- cyclic subspace = Smallest T- invariant subspace containing $\vec{v}$.

$\therefore W \subseteq X$. Clearly, $X \subseteq W$

Thus, $W = X$ and β = basis of W.

Also, $\dim(W) = k \Rightarrow j = k$.

② Let $a_0\vec{v} + a_1T(\vec{v}) + \dots + a_{k-1}T^{k-1}(\vec{v}) + T^k(\vec{v}) = 0$

$\Rightarrow T^k(\vec{v}) = -a_0\vec{v} - a_1T(\vec{v}) - \dots - a_{k-1}T^{k-1}(\vec{v})$

$$\therefore [T_W]_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & \vdots & \vdots \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix}$$

By M.I. and cofactor expansion on the first row, we get,

Char poly of T_W is: $f(t) = (-1)^k(a_0 + a_1t + \dots + a_{k-1}t^{k-1} + t^k)$

(Good exercise to check)

<u>Example 2</u>: Consider $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(a,b,c) = (2b+3c, 3a+8c, 8c)$

$T(e_1) = 3e_2$, $T^2(e_1) = 6e_1$ $\therefore$ T-cyclic subspace generated by $\vec{e_1}$ $= \text{span}\{\vec{e_1}, \vec{e_2}\}$

Now, $-6e_1 + T^2(e_1) = 0$.

$\therefore$ Char poly of T_W is: $f(t) = (-1)^2(-6 + t^2)$

<u>Check</u>: $\beta = \{\vec{e_1}, \vec{e_2}\}$ = basis of W.

$[T_W]_\beta = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$. $\therefore \det([T_W]_\beta - tI) = \det\begin{pmatrix} -t & 2 \\ 3 & -t \end{pmatrix} = t^2 - 6$.

Theorem 2: (Cayley – Hamilton) Let $T: V \to V$ (fin-dim).
$f(t)$ = char poly of T. Then: $f(T) = T_0$ = zero transformation)
(That is, T "satisfies" $f(t)$)

Proof: Need to show, $f(T)(\vec{v}) = \vec{0}$ for $\forall \vec{v} \in V$.
If $\vec{v} = \vec{0}$, obvious.
Suppose $\vec{v} \neq \vec{0}$. Let W = T-cyclic subspace generated by $\vec{v}$.
Suppose $\dim(W) = k$. So, $\exists\, a_0, a_1, \ldots, a_{k-1}$ such that
$a_0\vec{v} + a_1 T(\vec{v}) + \ldots + a_{k-1}T^{k-1}(\vec{v}) + T^k(\vec{v}) = \vec{0}$.
Thus, char poly of T_W is $(-1)^k(a_0 + a_1 t + \ldots + a_{k-1}t^{k-1} + t^k)$

$$\text{So, } g(T)(\vec{v}) = (-1)^k(a_0 I + a_1 T + \ldots + a_{k-1}T^{k-1} + T^k)(\vec{v})$$
$$= (-1)^k \underbrace{(a_0\vec{v} + a_1 T(\vec{v}) + \ldots + a_{k-1}T^{k-1}(\vec{v}) + T^k(\vec{v}))}_{\vec{0}}.$$

But $g(t)$ divides $f(t)$. Thus, $f(t) = q(t)\,g(t)$ for some $q(t)$.
So, $f(T)(\vec{v}) = q(T)g(T)(\vec{v}) = q(T)(\vec{0}) = \vec{0}$

Example 3: Let $T(a,b) = (3a+b,\ 2a+b)$. Let $\beta = \{\vec{e}_1, \vec{e}_2\}$ = ordered basis of $\mathbb{R}^2$.

Then: $A = [T]_\beta = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \rightsquigarrow$ Char poly of $T = t^2 - 4t + 1$

Easy to verify: $T^2 - 4T + I$ = zero transformation.
Also, $A^2 - 4A + I = \begin{pmatrix} 11 & 4 \\ 8 & 3 \end{pmatrix} - \begin{pmatrix} 12 & 4 \\ 8 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Remark: Cayley - Hamilton generally work for matrix.

Corollary 1: (Cayley-Hamilton for Matrices) Let $A \in M_{n\times n}(\mathbb{F})$.

Let $f(t)$ = char-poly of A. Then: $f(A) = O = n\times n$ zero matrix.

Invariant subspaces and Direct Sum

Theorem 3: Let $T: V \to V$ (fin-dim). Suppose that $V = W_1 \oplus \cdots \oplus W_k$

where W_i = T-invariant subspace of V for each i.

Let $f_i(t)$ = char. poly of T_{W_i}

Then: $f_1(t) f_2(t) \cdots f_k(t)$ = char poly of T.

Proof: We prove by M.I. on k. Let $f(t)$ = char poly of T.

When $k=2$, Let β_1 = ordered basis for W_1

β_2 = ordered basis for W_2.

Then: $\beta = \beta_1 \cup \beta_2$ = ordered basis for V. Let $B_1 = [T_{W_1}]_{\beta_1}$;

$B_2 = [T_{W_2}]_{\beta_2}$. Then: $A = [T]_\beta = \begin{pmatrix} B_1 & O \\ O & B_2 \end{pmatrix}$

$\therefore f(t) = \det(A - tI) = \det(B_1 - tI)\det(B_2 - tI) = f_1(t) f_2(t)$

$\therefore$ true for $k=2$.

Assume thm is valid for $k-1$ summand ($k-1 \geq 2$)

Let $V = W_1 \oplus \cdots \oplus W_k$. Let $W = W_1 + \cdots + W_{k-1}$

Then, W = T-invariant

Also, $V = W \oplus W_k$. So, $f(t) = g(t) f_k(t)$ where $g(t)$ = char poly of T_W.

Clearly, $W = W_1 \oplus W_2 \oplus \cdots \oplus W_{k-1}$.

$\therefore g(t) = f_1(t) f_2(t) \cdots f_{k-1}(t)$ by induction hypothesis.

$\therefore f(t) = g(t) f_k(t) = f_1(t) f_2(t) \cdots f_{k-1}(t) f_k(t)$.

<u>Remark</u>: Simple example: T = diagonalizable matrix.

$\lambda_1, \lambda_2, \ldots, \lambda_k$ = distinct eigenvalues.

$V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$. Then $f(t) = (\lambda_1 - t)^{m_1} \cdots (\lambda_k - t)^{m_k}$

Also, char poly of $T_{E_{\lambda_i}} = f_i(t) = (\lambda_i - t)^{m_i}$

$\therefore f(t) = f_1(t) \cdots f_k(t)$.

<u>Example 4</u>: Let $T(a,b,c,d) = (a+b, 2a-b, c+d, 3c+2d)$

Let $W_1 = \{(s,t,0,0) : s,t \in \mathbb{R}\}$; $W_2 = \{(0,0,s,t) : s,t \in \mathbb{R}\}$

Then: $\mathbb{R}^4 = W_1 \oplus W_2$. Let $\beta_1 = \{e_1, e_2\}$, $\beta_2 = \{e_3, e_4\}$.

Then: $[T_{W_1}]_{\beta_1} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$; $[T_{W_2}]_{\beta_2} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$

$$A = [T]_{\beta = \beta_1 \cup \beta_2} = \begin{pmatrix} \boxed{\begin{matrix} 1 & 1 \\ 2 & -1 \end{matrix}} & O \\ O & \boxed{\begin{matrix} 1 & 1 \\ 3 & 2 \end{matrix}} \end{pmatrix} = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

Then: $f(t) = \det(A - tI) = \det(B_1 - tI)\det(B_2 - tI)$

$$= \underbrace{f_1(t)}_{\text{char poly of } T_{W_1}} \cdot \underbrace{f_2(t)}_{\text{char poly of } T_{W_2}}$$

Definition 1: (Direct sum of Matrices)

Let $B_1 = M_{m\times m}(\mathbb{F})$, $B_2 = M_{n\times n}(\mathbb{F})$. We define the direct sum of B_1 and B_2, denote by $B_1 \oplus B_2$ as the $(m+n)\times(m+n)$ matrix A such that:

$$A_{ij} = \begin{cases} (B_1)_{ij} & 1 \le i,j \le m \\ (B_2)_{(i-m)\times(j-m)} & m+1 \le i,j \le n+m \\ 0 & \end{cases} \qquad A = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

If $B_1, B_2, \ldots, B_k$ are square matrices, we define the direct sum of $B_1, B_2, \ldots, B_k$ recursively by:

$$B_1 \oplus B_2 \oplus \cdots \oplus B_k = (B_1 \oplus \cdots \oplus B_{k-1}) \oplus B_k.$$

$$A = \begin{pmatrix} \boxed{B_1} & & & \\ & \boxed{B_2} & & 0 \\ & & \ddots & \\ & 0 & & \boxed{B_k} \end{pmatrix}$$

Example 4: Let $B_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$; $B_2 = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$; $B_3 = (1)$

Then: $B_1 \oplus B_2 \oplus B_3 = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Theorem 4: Let $T: V \to V$ (fin-dim). Let $W_1, \ldots, W_k =$ T-invariant subspaces of V such that $V = W_1 \oplus \cdots \oplus W_k$. Let $\beta_i =$ ordered basis for W_i. Let $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$.

Let $A = [T]_\beta$ and $B_i = [T_{W_i}]_{\beta_i}$

Then: $A = B_1 \oplus B_2 \oplus \cdots \oplus B_k$