Lecture 1: Revision (1)

Definition of vector space

A vector space over a field $\mathbb{F}$ (can be $\mathbb{R}$ or $\mathbb{C}$) is a set V with two operations (addition $+$ and scalar multiplication $\cdot$) such that:

(a) if $\vec{v}, \vec{w} \in V$, then $\vec{v} + \vec{w} \in V$
(b) if $a \in \mathbb{F}$ and $\vec{v} \in V$, then $a \cdot \vec{v} \in V$

Closed under $+$ and $\cdot$

Also, it satisfies 8 properties

(VS1): $\vec{x} + \vec{y} = \vec{y} + \vec{x} \quad \forall \vec{x}, \vec{y} \in V$

(VS2): $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z}) \quad \forall \vec{x}, \vec{y}, \vec{z} \in V$

(VS3): $\exists\, \vec{0} \in V$ s.t. $\vec{x} + \vec{0} = \vec{x} \quad \forall \vec{x} \in V$

(VS4): $\forall \vec{x} \in V$, $\exists\, \vec{y} \in V$ s.t. $\vec{x} + \vec{y} = \vec{0}$

(VS5): $1 \cdot \vec{x} = \vec{x} \quad \forall \vec{x} \in V$

(VS6): $(ab) \cdot \vec{x} = a \cdot (b \cdot \vec{x}) \quad \forall a, b \in \mathbb{F}, \forall \vec{x} \in V$

(VS7): $a \cdot (\vec{x} + \vec{y}) = a \cdot \vec{x} + a \cdot \vec{y} \quad \forall a \in \mathbb{F}, \forall \vec{x}, \vec{y} \in V$

(VS8): $(a+b) \cdot \vec{x} = a \cdot \vec{x} + b \cdot \vec{x} \quad \forall a, b \in \mathbb{F}, \forall \vec{x} \in V$

(Please refer to p.7 in the textbook)

Examples of vector spaces:

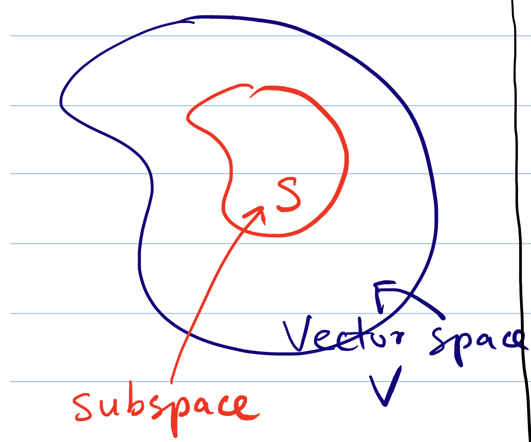
(a) $\mathbb{R}^n$, $\mathbb{C}^n$; (b) $M_{m\times n}(\mathbb{R})$ (m×n real matrices)

(c) $P_n(\mathbb{R}) := \{$ real polynomials of degree $\leq n \}$

(d) $C(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R}$ continuous $\}$ (infinite dimensional)

(e) $P(\mathbb{R}) := \{$ polynomials over $\mathbb{R} \}$

Subspaces: Vector space which is a subset of a vector space (over the same operations)



Thm: W is a subspace of V over $\mathbb{F}$ if and only if:

(1) $\vec{0}_V \in W$

(2) If $\vec{x}, \vec{y} \in W$, then: $\vec{x} + \vec{y} \in W$

(3) If $c \in \mathbb{F}$, $\vec{x} \in W$, then: $c \cdot \vec{x} \in W$

(Please refer to P.17 in the textbook)

Examples of subspaces:

(a) $\{\vec{0}\}$, V — trivial subspaces of V

(b) $\{\text{Symmetric matrices}\} \subseteq M_{n\times n}(\mathbb{F})$

(c) $P_n(\mathbb{R}) \subseteq P(\mathbb{R})$

Thm: W_1 and W_2 subspaces $\Rightarrow$ $W_1 \cap W_2$ subspace

Thm: W_1 and W_2 subspaces $\Rightarrow$ $W_1 + W_2$ subspace

$$W_1 + W_2 = \{\vec{w}_1 + \vec{w}_2 : \vec{w}_1 \in W_1, \vec{w}_2 \in W_2\}$$

[Note: W_1 and W_2 subspaces $\not\Rightarrow$ $W_1 \cup W_2$ subspace]

Span: For any subset $S \subseteq V$, the span of S is defined as:

$$\text{Span}(S) = \{ \underbrace{\sum_{i=1}^{n} \lambda_i \cdot \vec{w}_i}_{\text{Linear combination of vectors of } S} : \lambda_i \in \mathbb{F}, \vec{w}_i \in S \}$$

Linearly dependent: A subset S is linearly dependent if $\exists\ \lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{F}$ (not all 0) and distinct $\vec{w}_i \in S$ s.t. $\lambda_1 \vec{w}_1 + \lambda_2 \vec{w}_2 + \cdots + \lambda_k \vec{w}_k = \vec{0}$.

<u>Linearly independent</u> : S is linearly independent if S is not linearly dependent.

<u>Thm</u>: S is linearly independent iff :

" $\lambda_1 \vec{w}_1 + \dots + \lambda_k \vec{w}_k = \vec{0}$ " $\Rightarrow$ " $\lambda_1 = \lambda_2 = \dots = \lambda_k = 0$ "
$\lambda_i \in \mathbb{F}$, $\vec{w}_k \in S$

<u>Thm</u>:

- $\underbrace{S_1}_{\text{lin dep}} \subseteq S_2 \Rightarrow S_2$ lin. dep
- $S_1 \subseteq \underbrace{S_2}_{\text{lin indep}} \Rightarrow S_1$ lin. indep.
- If $S \subseteq V$ is lin. ind, then:
  $S \cup \{\vec{v}\}$ lin dep $\Longleftrightarrow$ $\vec{v} \in \mathrm{Span}(S)$

<u>Basis of V</u> : $\mathrm{Span}(\beta) = V$ and β is linearly independent

<u>Dimension</u>: V is called finite dimensional if $\exists$ finite basis β. Dimension of V :
$\dim(V) = \#$ of elements in β
Otherwise, V is infinite dimensional

<u>Examples of basis</u> : (a) $\beta = \{1, x, x^2, \dots, x^n\}$ is a basis of $P_n(\mathbb{R})$

(b) $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ is a basis of $\mathbb{R}^n$ where
$\vec{e}_i = (0, 0, \dots, 0, \underset{\uparrow\, i\text{th entry}}{1}, 0, \dots, 0)$

Thm: Let V be a n-dimensional vector space over $\mathbb{F}$, then:

(a) S lin. indep subset $\Rightarrow |S| \leq n$ (# of elements in S)

(b) $\mathrm{Span}(S) = V \Rightarrow |S| \geq n$

(c) Any linearly indep subset S of V can be extended to a basis of V

(d) $W \subseteq V$ subspace $\Rightarrow \dim(W) \leq \dim V$
"=" holds iff $W = V$.

(Please refer to P.39 in the textbook)

Remark: With a (ordered) basis $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ of a vector space V, V can be "identified" with $\mathbb{F}^n$:

For each element $\vec{v} \in V$, write $\vec{v} = \lambda_1 \vec{v}_1 + \dots + \lambda_n v_n$.
$\vec{v}$ can be "identified" by: $[\vec{v}]_\beta := \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}$.

($\lambda_1, \lambda_2, \dots, \lambda_n$ is uniquely determined given β)
$[\vec{v}]_\beta$ is called the coordinate vector w.r.t. β.