25/9/2014 Math 1010C Tutorial-3 Jiang Qingyuan

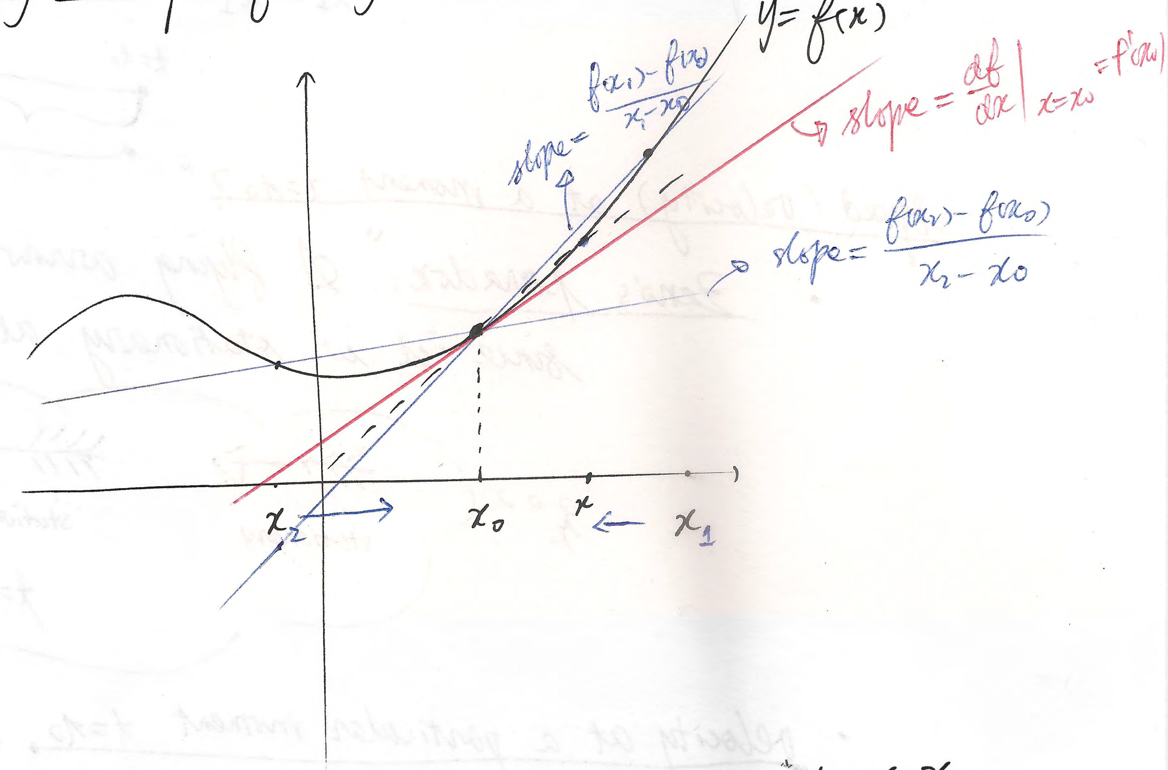
- Derivatives: WHAT is it & WHY it is useful;
- Practice with limits & computation of derivatives;

Derivatives: $y = f(x)$, its derivatives at x_0 is defined if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \quad \exists$$

& denote this number $y'(x_0) = f'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0} = \left.\frac{df}{dx}\right|_{x=x_0}$;

- Geometric meaning: slope of tangent line at $x = x_0$:



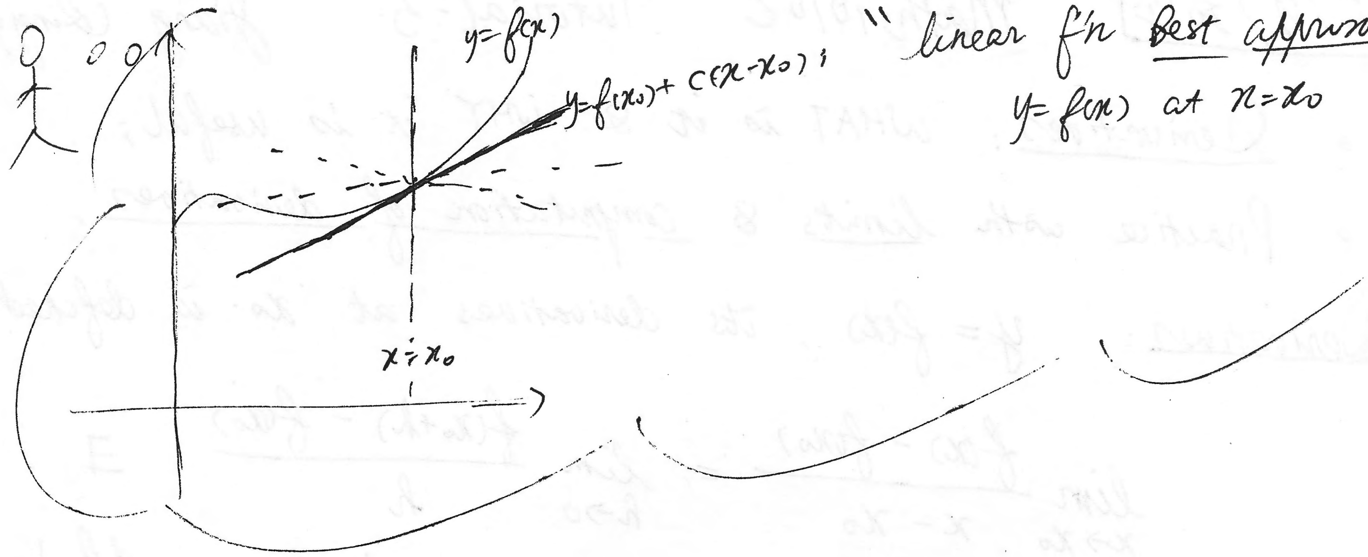
- Interpretation II: linearization of function at a point $x = x_0$;

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \quad \text{as } x \text{ near } x_0.$$

More precisely. $f'(x_0) = C$ if and only if $\exists$ fn $\alpha(x)$ s.t

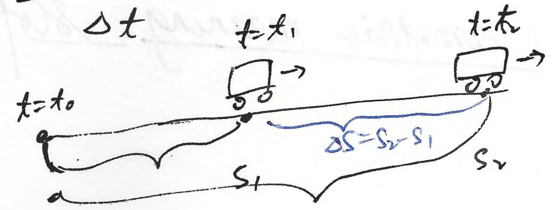
$$\begin{cases} f(x) = f(x_0) + C(x - x_0) + \underbrace{\alpha(x-x_0)}_{\text{often denoted } o(x-x_0) \text{ (OR sometimes \& under certain conditions } O((x-x_0)^2))}; \\ \lim_{h \to 0} \frac{\alpha(h)}{h} \text{ exists and } = 0; \end{cases}$$

" linear f'n best approximate $y=f(x)$ at $x=x_0$ "



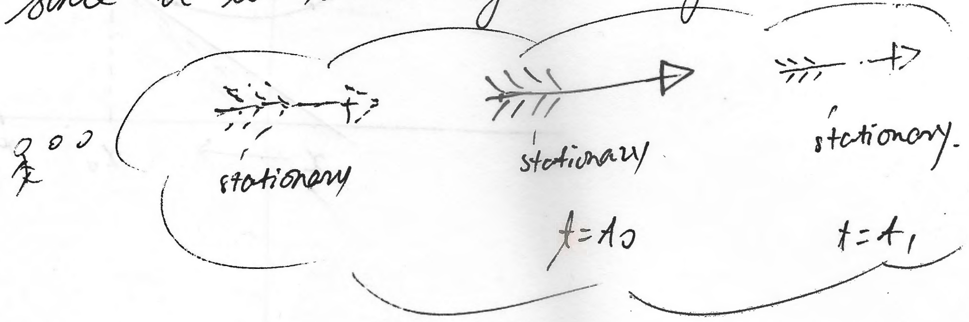
Understanding Derivatives: Speed (or velocity) at a moment.

Average speed: $\bar{v} = \dfrac{S_2 - S_1}{t_2 - t_1} = \dfrac{\Delta S}{\Delta t}$



Speed (velocity) at a moment $t=t_0$?

- Zeno's paradox: " A flying arrow does not move, since it is stationary at every moment !"



- velocity at a particular moment $t=t_0$, is defined to be

$$v\big|_{t=t_0} := \lim_{t\to t_0} \frac{S(t)-S(t_0)}{t-t_0} = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t};$$

measures the tendency of how "fast", how much distance the "arrow" ~~oth~~ will travel at that particular moment $(t=t_0)$ as time goes on.

Modern answer to Zeno's paradox:
Motion is possible, since although the arrow is stationary at every particular moment $t=t_0$, but the velocity at that moment is not zero ($\frac{dS}{dt}\big|_{t=t_0} = v(t=t_0) \neq 0$).

WHY derivatives are useful ?

Supplementary Example. Not Compulsory.

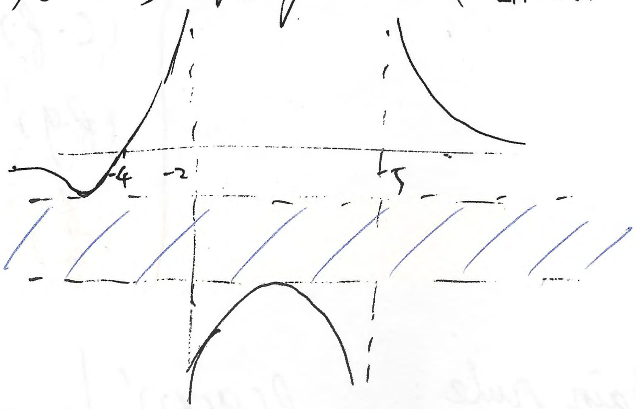
Illustrate with an example, one exercise from tutorial - 1, namely, we ask you to compute the range of the f'n

$$f(x) = \frac{x+4}{x^2-3x-10} \; ; \quad (\text{\& try to draw its rough picture}).$$

we use let $f(x) = \frac{1}{a}$ & $\Delta \geq 0 \Rightarrow$ Range: $\mathbb{R} \setminus \left(\frac{1}{-11+6\sqrt{2}}, -\frac{1}{11+6\sqrt{2}} \right)$;

& picture roughly

(see next page for recalling this e.g.)



Now we use our new tool derivatives to deduce this again:

$$f(x) = \frac{x+4}{x^2-3x-10} \; ;$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - f\cdot g'}{g^2}$$

$$f'(x) = \frac{1\cdot(x^2-3x-10) - (x+4)(2x-3)}{(x^2-3x-10)^2} \; ;$$

$$= -\frac{x^2+8x-2}{(x^2-3x-10)^2} \; ;$$

Solve $x^2+8x-2=0 \Leftrightarrow (x+4)^2 = 18 \Leftrightarrow x = -4 \pm 3\sqrt{2}$;

& compute $f(-4+3\sqrt{2}) = \frac{3\sqrt{2}}{-11(-4+3\sqrt{2})-8} = \frac{3\sqrt{2}}{36-33\sqrt{2}} = \frac{1}{6\sqrt{2}-11}$;

$x^2+8x-2=0 \Leftrightarrow x^2-3x-10 = -11x-8$

$f(-4-3\sqrt{2}) = \frac{-3\sqrt{2}}{-11(-4-3\sqrt{2})-8} = \frac{-3\sqrt{2}}{36+33\sqrt{2}} = \frac{-1}{6\sqrt{2}+11}$;

x	$-\infty$	$(-\infty, -4-3\sqrt{2})$	$-4-3\sqrt{2}$	$(-4-3\sqrt{2}, -2)$	-2	$(-2, -4+3\sqrt{2})$	$-4+3\sqrt{2}$	$(-4+3\sqrt{2}, 5)$	5	$(5,\infty)$
$f'(x)$		<0	0	>0	//	>0	0	<0	//	<0
$f(x)$	0	↘	$-\frac{1}{6\sqrt{2}+11}$	↗ $+\infty$		$-\infty$ ↗	$\frac{1}{6\sqrt{2}-11}$	↘		↘ 0

11/9/2014

Recall. From Tutorial-1.

1 (a) $f(x) = \dfrac{x+4}{x^2-3x-10}$;

maximal domain is $\mathbb{R} \setminus \{-2, 5\}$;

for the range, · f can achieve 0: $x = -4$;

· Can f achieve $a \neq 0$?

i.e. $\exists ?\ x \in \mathbb{R} \setminus \{-2,5\}$, s.t. $\dfrac{x+4}{x^2-3x-10} = a$.

$$\Longleftrightarrow \quad x^2 - 3x - 10 = \frac{x}{a} + \frac{4}{a}$$

$$\Longleftrightarrow \quad x^2 - \left(3 + \frac{1}{a}\right) x - 10 - \frac{4}{a} = 0$$

this has solution iff

$$\Delta = \left(3 + \frac{1}{a}\right)^2 - 4\left(-10 - \frac{4}{a}\right)$$

$$= \frac{1}{a^2} + \frac{22}{a} + 49$$

Recall. From Tutorial-1.

$$= \left(\frac{1}{a} + 11\right)^2 + \underbrace{49 - 121}_{(-72)} \geq 0.$$

$$\Longleftrightarrow \quad \left|\frac{1}{a} + 11\right| \geq \sqrt{72} = 6\sqrt{2};$$

$$\Longleftrightarrow \quad \frac{1}{a} + 11 \geq 6\sqrt{2} \text{ or } \frac{1}{a} + 11 \leq -6\sqrt{2};$$

$$\Longleftrightarrow \quad a \in \left(-\infty, \frac{1}{-11+6\sqrt{2}}\right] \cup \left[-\frac{1}{11+6\sqrt{2}}, 0\right) \cup (0, +\infty)$$

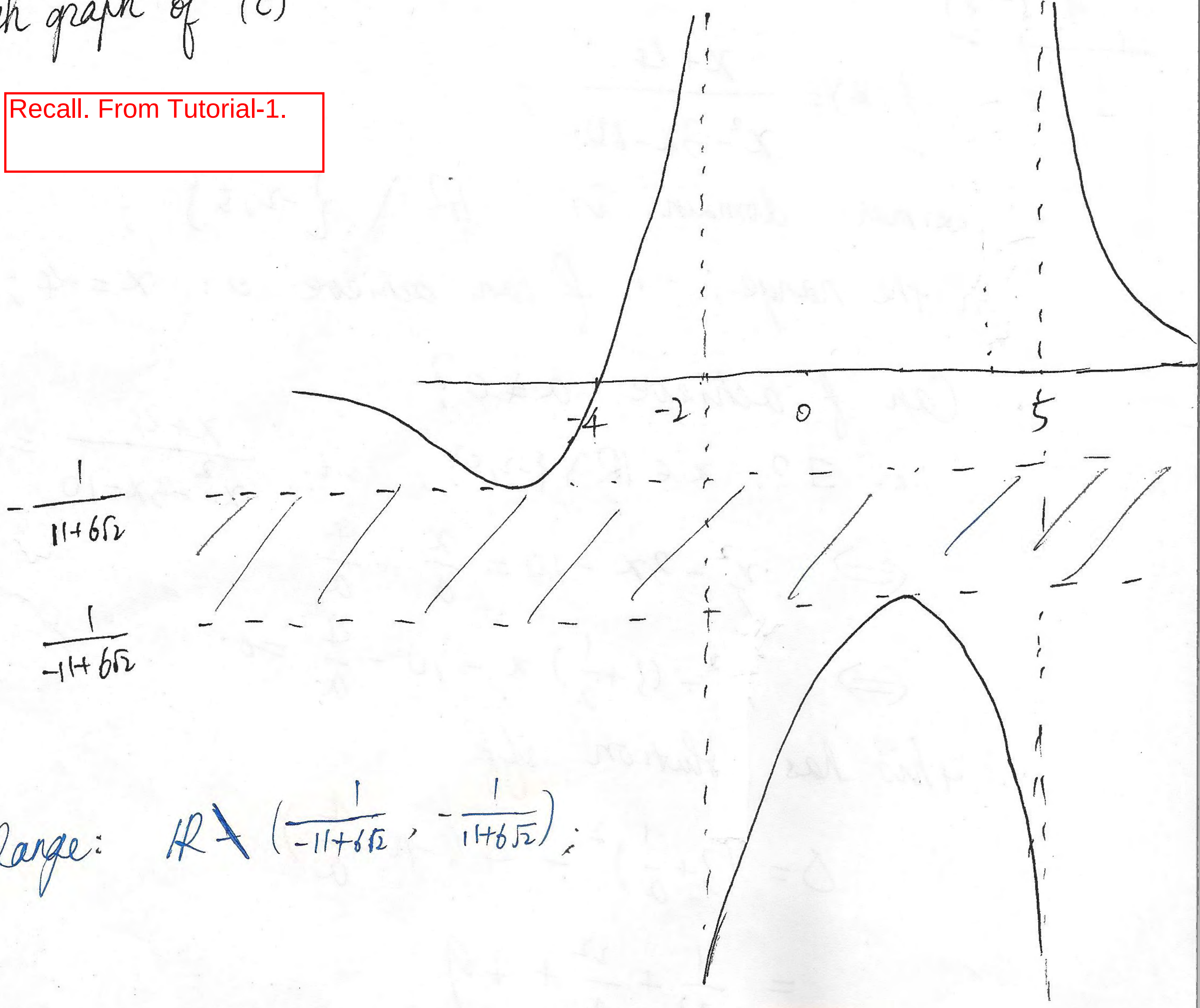
Hence the range is $\quad -\dfrac{11+6\sqrt{2}}{49} \quad \dfrac{-11+6\sqrt{2}}{49}$

$$f(\mathbb{R} \setminus \{-2,5\}) = \mathbb{R} \setminus \left(\frac{1}{-11+6\sqrt{2}}, -\frac{1}{11+6\sqrt{2}}\right) \quad \square$$

P1+1

Rough graph of (c)

Recall. From Tutorial-1.



Range: $\mathbb{R} \setminus \left(\frac{1}{-11+6\sqrt{2}}, -\frac{1}{11+6\sqrt{2}}\right)$.

$\square$ of 1(c).

Recall. From Tutorial-1.

Hence agrees with the graph, & KNOWN: at which pt it reach local maximum or minimum;

(Review)

Compute derivatives: Basic Rules:

· Algebra of derivatives:
$$\begin{cases} (f \pm g)' = f' \pm g'; \\ (C \cdot f)' = C \cdot f', \quad C = \text{const}; \\ (f \cdot g)' = f' \cdot g + f \cdot g'; \\ \left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}; \end{cases}$$

· Chain rule: $f(g(x))'\big|_{x=x_0} = f'(g(x_0)) \cdot g'(x_0)$;

(Review)

Rules of limits

(H1) $\lim_{x \to c} \phi(x) = A$, $\lim_{x \to c} \phi(x) = B \Rightarrow A = B$ (uniqueness)

$\odot = +, -, \times$;

(H2) $\lim_{x \to c} (\phi \odot \eta)(x) = A \odot B$ provided $\lim_{x \to c} \phi(x) = A$, $\lim_{x \to c} \eta(x) = B$ & when $\odot = \div$, $B \neq 0$; (arithmetics)

(H3) $\lim k\eta(x) = k \cdot \lim \eta(x)$;

(H4) $f(x) \leq g(x) \Rightarrow \lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$ provided they exist; (Comparison)

(H5) $f(x) \leq g(x) \leq h(x)$ & $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} f(x)$ exists, & $= L$. (Sandwich thm)

Punctured interval lemma: $f(x) = g(x)$, $\forall x \in (a,b) \setminus \{c\}$; $c \in (a,b)$

Then $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$;

Supplementary Exercise : Chapter 2: Derivatives

1. Evaluate the following limits:

(a) $\lim_{x\to 2}(x^3-5x+4)$; (d). $\lim_{x\to 2}\dfrac{x^2+3x-10}{x^2-x-2}$;

(e) $\lim_{x\to 4}\dfrac{2-\sqrt{x}}{4-x}$; (k): $\lim_{t\to 0}\left(\dfrac{1}{2t}-\dfrac{1}{t^2+2t}\right)$;

3. Use definition to evaluate the derivatives of the following f'ns:

(a) $y=3x-2$; (c): $y=x^4$;

(h)* $y=\cos x$; (j)*: $y=e^x$;

5. Find the (first) derivatives of the following functions:

(a) $y=x^3-4x+3$, (b) $y=\sqrt{x}+\dfrac{1}{\sqrt{x}}$;

(h) $y=\dfrac{3x-4}{x+2}$; (l) $y=(x^2+1)^7$;

(n)* $y=\cos(x^2)$; (p)* $y=\ln(\ln x)$;

$\boxed{\text{Think about}}$: How can one compute the derivatives of the following f'ns:

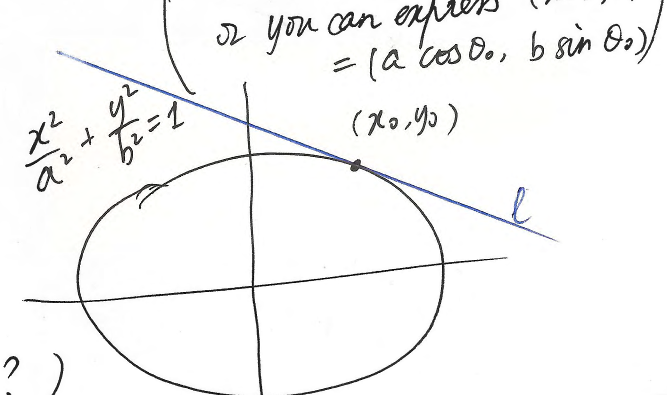
(I). $y=x^x$; (& then $y=x^{x^x}$?)

talk about these two next time

(II). Consider the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$; $(a>b>0)$

& a point (x_0, y_0) on it. ($\dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}=1$; or you can express $(x_0,y_0)=(a\cos\theta_0,\ b\sin\theta_0)$)

Can you compute the slope of the tangent line at (x_0,y_0) ?



(& further, compute the equation of the tangent line l ?)

Solutions:

1. (a). $\lim_{x\to 2}(x^3-5x+4)$

Use (H2): $\lim_{x\to 2}(x\cdot x\cdot x - 5\cdot x+4) = 2\cdot 2\cdot 2 - 5\cdot 2+4$

$= 8-10+4 = 2$; $\square$

(d) $\lim_{x\to 2}\frac{x^2+3x-10}{x^2-x-2}$

Use (H2)? : $\frac{2^2+3\cdot 2-10}{2^2-2-2} = \frac{10-10}{4-2-2} = \frac{0}{0} \Rightarrow$ can not use (H2).

But $\frac{x^2+3x-10}{x^2-x-2} = \frac{\cancel{(x-2)}(x+5)}{\cancel{(x-2)}(x+1)} = \frac{x+5}{x+1}$

Now use (H2)

$\lim_{x\to 2}(\cdots) = \frac{2+5}{2+1} = \frac{7}{3}$; $\square$

(e). $\lim_{x\to 4}\frac{2-\sqrt{x}}{4-x}$

can not use (H2) directly. But $\frac{2-\sqrt{x}}{4-x} = \frac{\cancel{(2-\sqrt{x})}}{\cancel{(2-\sqrt{x})}(2+\sqrt{x})} = \frac{1}{2+\sqrt{x}}$

then the limit is $\frac{1}{2+\sqrt{4}} = \frac{1}{4}$; $\square$

(k) $\lim_{t\to 0}\left(\frac{1}{2t}-\frac{1}{t^2+2t}\right)$

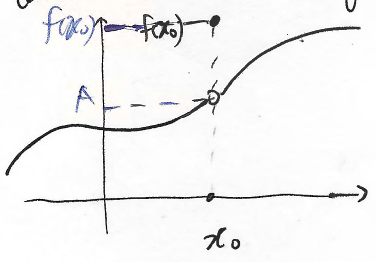
$(\infty-\infty)$, can not use (H2) directly.

But $\frac{1}{2t}-\frac{1}{t^2+2t} = \frac{t^2+2t-2t}{2t(t^2+2t)} = \frac{\cancel{t^2}}{2\cancel{t}\cdot\cancel{t}\cdot(t+2)} = \frac{1}{2(t+2)}$

then use (H2), get $\frac{1}{4}$; $\square$

Rmk: $\lim_{x\to x_0} f(x)$ has nothing to do with the value $f(x_0)$.

(Punctured interval lemma):



$\lim_{x\to x_0} f(x)$ is still A.

3. Use def'n of derivatives to compute " of :

(a) $y = 3x-2$;

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{(3x-2)-(3x_0-2)}{x-x_0} = \lim_{x\to x_0} \frac{3\cancel{(x-x_0)}}{\cancel{x-x_0}}$$

$= 3$;

□

(c) $y = x^4$;

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{x^4-x_0^4}{x-x_0}$$

since $x^4 - x_0^4 = (x-x_0)(x^3+x^2x_0+x\cdot x_0^2+x_0^3)$

Hence use (H2), $f'(x_0) = x_0^3 + x_0^2\cdot x_0 + x_0\cdot x_0^2 + x_0^3 = 4x_0^3$;

$\Rightarrow f'(x) = 4x^3$;

□

(h)* $y = \cos x$;

Method I. text book (2.8). $\cos(x+h) = \cos h\cos x - \sin h \sin x$;

$$(\cos x)' = \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0} \frac{(\cos h - 1)\cos x}{h} - \lim_{h\to 0}\frac{\sin h}{h}\sin x$$

$$= 0\cdot\cos x - 1\cdot\sin x = -\sin x;$$

& use geometric picture, or for $0 < h < \frac{\pi}{2}$

$$\begin{cases} h - \frac{h^3}{6} < \sin h < h \Rightarrow \lim_{h\to 0}\frac{\sin h}{h} = 1; \\ -\frac{h^2}{2} < \cos h - 1 < -\frac{h^2}{2} + \frac{h^4}{24} \Rightarrow \lim_{h\to 0}\frac{\cos h - 1}{h} = 0; \end{cases}$$

Method II (formally). $y = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{6!} + \cdots$

$$y' = -x + \frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \cdots$$

$$= -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= -\sin x;$$

□

3. (j)* $y = e^x$;

Method I: use $e^{x+h} = e^x \cdot e^h$; $\dfrac{e^{x+h} - e^x}{h} = e^x \dfrac{e^h - 1}{h}$;

then $e^h = 1 + h + \frac{h^2}{2} + \cdots$ $\dfrac{e^h - 1}{h} = 1 + \frac{h}{2} + \cdots \to 1$;

$$\Rightarrow \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = e^x;$$

Method II. $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$(e^x)' = 1 + x + \frac{x^2}{2!} + \frac{x^4}{3!} + \cdots = e^x$;

$\square$

5. (a) $y = x^3 - 4x + 3$;

$y' = 3x^2 - 4$; $\square$

(b) $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$ $(x > 0)$, use $y = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$ then

$$y' = \frac{1}{2} \cdot x^{\frac{1}{2}-1} + \left(-\frac{1}{2}\right) \cdot x^{-\frac{1}{2}-1} = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}};$$

$\left(\text{OR} \quad \left(\dfrac{1}{\sqrt{x}}\right)' = \dfrac{0 \cdot \sqrt{x} - \frac{1}{2\sqrt{x}}}{(\sqrt{x})^2} = -\dfrac{1}{2(\sqrt{x})^3}; \right)$ $\square$

(h) $y = \dfrac{3x - 4}{x + 2}$;

then $y' = \dfrac{3 \cdot (x+2) - (3x-4) \cdot 1}{(x+2)^2} = \dfrac{10}{(x+2)^2}$;

OR $y = \dfrac{3x + 6 - 10}{x + 2} = 3 - \dfrac{10}{x+2} \Rightarrow y' = \dfrac{10}{(x+2)^2}$; $\square$

(l) $y = (x^2 + 1)^7$; use chain rule

$y' = 7(x^2+1)^6 \cdot (x^2+1)' = 14(x^2+1)^6 x$; $\square$

(n)* . $y = \cos(x^2)$;

$$y' = -\sin(x^2) \cdot 2x = -2x\sin(x^2) \; ; \qquad \square$$

(p)* . $y = \ln(\ln x)$;

$$y' = \frac{1}{\ln x} \cdot (\ln x)' = \frac{1}{x\ln x} \; ; \qquad \square$$

<u>think about!</u>

(I) $y = x^x$. then ... tell you next time ...

(II) . //

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