MATH 1510E_H Notes

Definite Integral & FTC

Topics covered

- Riemann Sum
- Fundamental theorem of calculus
- Applications

Until now, when we talked about integral, we mean "indefinite integral" or the solutions to the differential equation $F'(x) = f(x)$.

We have denoted such integrals by the symbol $\int f(x)dx$.

We also noticed that $\int f(x)dx$ and $\int f(x)dx + C$ are both solutions to the differential equation $F'(x) = f(x)$.

But "integration" has another meaning. It is the "computation" of "area" under the curve $y = f(x), a \le x \le b$.

Q: How to define this kind of integral? What is its name?
A: It is called definite integral and is defined as follows:

Suppose we have a continuous function $f: [a, b] \to \mathbb{R}$ and we want to compute the "area" under the curve $y = f(x)$, for $x \in [a, b]$. The we can do this by the following

Method to find Area "under" a curve:

(Step 1) Partition the interval $[a, b]$ into n subintervals defined by the points

$$a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$$

This way, we have n subintervals, i.e. $[x_0, x_1], [x_1, x_2], \cdots, [x_{i-1}, x_i], \cdots, [x_{n-1}, x_n]$.

(Step 2) Define by the symbol $\|P\|$ and call it "length" of P by letting

$$\|P\| = maximum\ among\ x_1 - x_0, x_2 - x_1, \cdots, x_i - x_{i-1}, \cdots, x_n - x_{n-1}$$

Therefore, if $\|P\| \to 0$, then all the numbers $x_1 - x_0, x_2 - x_1, \cdots, x_i - x_{i-1}, \cdots, x_n - x_{n-1}$ will go to zero.

(Step 3) Construct n rectangles "under" the curve $y = f(x)$, by choosing as heights the numbers $f(\xi_i)$, where ξ_i is any number between x_{i-1} and x_i. Choose widths to be the numbers $x_i - x_{i-1}$.

Such rectangles have then areas equal to $f(\xi_i) \cdot (x_i - x_{i-1})$
The sum of these areas is then equal to

$$\sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1})$$

or equal to

$$\sum_{i=1}^{n} f(\xi_i) \cdot \Delta x_i$$

if we let $\Delta x_i = x_i - x_{i-1}$.

(Step 4) Now one can show (with more mathematics) that for continuous function f, as $\|P\| \to 0$, the following limit is always a finite number:

$$\lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\xi_i) \cdot \Delta x_i$$

(Step 5) Finally, we give a symbol to this limit and call it $\int_a^b f(x)dx$.

In conclusion, we have (for continuous function $f: [a,b] \to \mathbb{R}$) the following:

$$\lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\xi_i) \cdot \Delta x_i = \int_a^b f(x)dx.$$

Remarks

- This kind of sum are called Riemann sums
- It can be shown that $\|P\| \to 0 \; implies \; n \to \infty$

This limit, $\int_a^b f(x)dx$ is called the "definite integral" of f for $a \leq x \leq b$.

Example

Consider the function $f(x) = x$, for $0 \leq x \leq 1$.

Partition $[0,1]$ into n subintervals of the form:

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \cdots, \left[\frac{i-1}{n}, \frac{i}{n}\right], \cdots, \left[\frac{n-1}{n}, \frac{n}{n}\right]$$

Each of these subintervals has length $\frac{1}{n}$, therefore $\|P\| = \frac{1}{n}$, which means as

$\|P\| = \frac{1}{n} \to 0$, it follows that $n \to \infty$.

Next, consider the following sum of areas of rectangles, where we choose $\xi_i = x_i = \frac{i}{n}$, then we have the sum

$$\sum_{i=1}^{n} f(x_i)\cdot\Delta x_i = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\cdot\frac{1}{n} = \sum_{i=1}^{n}\frac{i}{n}\cdot\frac{1}{n} = \sum_{i=1}^{n}\frac{i}{n^2}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} i = \frac{1}{n^2}\cdot\frac{(1+n)n}{2} = \frac{n+1}{2n} = \left(\frac{1}{2}\right)\left(1+\frac{1}{n}\right)$$

Hence, as $\|P\| \to 0$, it follows that $n \to \infty$ and also $\lim_{\|P\|\to 0}\sum_{i=1}^{n} f(x_i)\cdot\Delta x_i =$

$\frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=\frac{1}{2}.$

Remark: The choice of the points ξ_i is arbitrary. One can choose (i) the left endpoint, (ii) the right endpoint, (iii) the mid-points, (iv) the absolute maximum points, (v) the absolute minimum points etc.

No matter what one chooses for ξ_i, the limit remains the same.

Properties of Definite Integrals

The following properties of definite integrals are consequences of the area of a rectangle.

1. $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

2. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

4. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

One also has the following simple inequality (which hasn't been mentioned in the lectures) as well as Mean Value Theorem.

5. If $f(x) \leq g(x)$, $a \leq x \leq b$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

6. $\int_a^b f(x)dx = f(\xi)(b-a), \exists \xi \in [a,b]$

Remarks

- The mean value theorem here uses closed interval $[a,b]$.
- Using the above-mentioned Riemann Sum method to find area under a curve $y = f(x), a \leq x \leq b$ is very tedious. There is a more effective method, which computes area by (i) first compute an indefinite integral $F(x) = \int f(x)dx + C$, then (ii) compute the number $F(b) - F(a)$. This number is the the area wanted. This method is called the Fundamental Theorem of Calculus (FTC) outlined below.
- This FTC method doesn't always work. For some functions, such as $f(x) = e^{x^2}$, one cannot find a "closed form" function $F(x) = \int e^{x^2}dx + C$. For such functions $f(x)$, the areas have to computed using other methods, such as the Riemann sum.

Fundamental Theorem of Calculus (FTC)

There are two parts in the Fundamental Theorem of Calculus (in the future, we just write "FTC" for it).

(Part I of FTC)

Let $f(x)$ be a continuous function defined on the closed interval $[a,b]$. Then the following holds

$$\frac{d\int_a^x f(t)dt}{dx} = f(x)$$

for each $x \in (a,b)$.

(**Terminology:** We call this function $\int_a^x f(t)dt$ the "area-finding function". This function computes the area "under" the curve $y = f(t)$ for those t from a to x.)

(Part II of FTC)

For any solution $F(x)$ which satisfies the "differential" equation

$$F'(x) = f(x) \text{ for } x \in (a,b),$$

we can compute the area under the curve $y = f(x)$ for $a \leq x \leq b$, by the formula

$\int_a^b f(t)dt = F(b) - F(a)$

Note that one can use any symbol, e.g. x, u instead of t here. I.e.

$$\int_{x=a}^{x=b} f(x)dx = \int_{u=a}^{u=b} f(u)du = \int_{t=a}^{t=b} f(t)dt = F(b) - F(a)$$

Further F.T.C. (Fundamental Theorem of Calculus)

One can widely extend the FTC to compute things like the following:

$$\frac{d}{dx}\int_{t=a(x)}^{t=b(x)} f(x,t)dt$$

Goal: We want to show that (in the following, for simplicity, we omit the variable t in the lower and upper sum of the integral).

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x}dt$$

Proof:

(Main Idea): View $\int_{a(x)}^{b(x)} f(x,t)dt$ this way.

Suppose instead of $\int_{a(x)}^{b(x)} f(x,t)dt$, we consider the expression $\int_A^B f(C,t)dt$ and think of it as a "function of 3 variables A, B and C").

Let's denote this function by $g(A,B,C)$.

(Step 1) For a function $g(A,B,C)$ of several variables, say 3 variables, we have the following Chain Rule (if $A = a(x), B = b(x), C = x$):

$\frac{dg(A,B,C)}{dx} = \frac{dg(a(x),b(x),x)}{dx} = g_1(a(x),b(x),c(x)) \cdot a'(x) + g_2(a(x),b(x),c(x)) \cdot b'(x) + g_3(a(x),b(x),x) \cdot 1$

Remark: $g_1(a(x),b(x),x)$ means "$\frac{\partial g(A,B,C)}{\partial A}$ evaluated at the point $A = a(x), B =$

$b(x), C = x$". Similarly, $g_2(a(x), b(x), x)$ means "$\frac{\partial g(A,B,C)}{\partial B}$ evaluated at the point $A = a(x), B = b(x), C = x$", $g_3(a(x), b(x), x)$ means "$\frac{\partial g(A,B,C)}{\partial C}$ evaluated at the point $A = a(x), B = b(x), C = x$"

(Step 2) We apply this Chain Rule to our function of 3 variables $\int_A^B f(C,t)dt$ and obtain

$$\frac{d}{dx}\int_A^B f(C,t)dt = \frac{\partial \int_A^B f(C,t)dt}{\partial A}\cdot\frac{dA}{dx} + \frac{\partial \int_A^B f(C,t)dt}{\partial B}\cdot\frac{dB}{dx} + \frac{\partial \int_A^B f(C,t)dt}{\partial C}\cdot\frac{dC}{dx}$$

Remark: In the above formula, we wrote $\frac{dA}{dx}, \frac{dB}{dx}, \frac{dC}{dx}$ because A, B, C are functions of one variable x, so there is no need to use $\frac{\partial}{\partial x}$!

Now, the formula

$$\frac{d}{dx}\int_A^B f(C,t)dt = \frac{\partial \int_A^B f(C,t)dt}{\partial A}\cdot\frac{dA}{dx} + \frac{\partial \int_A^B f(C,t)dt}{\partial B}\cdot\frac{dB}{dx} + \frac{\partial \int_A^B f(C,t)dt}{\partial C}\cdot\frac{dC}{dx}$$

is the same as

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = \frac{\partial(-\int_B^A f(C,t)dt)}{\partial A}\cdot\frac{da(x)}{dx} + \frac{\partial \int_A^B f(C,t)dt}{\partial B}\cdot\frac{dB}{dx} + \frac{\partial \int_A^B f(C,t)dt}{\partial C}\cdot\frac{dx}{dx}$$

because $A = a(x), B = b(x), C = x$.

Now we use FTC (the usual FTC) to get $\frac{\partial(-\int_B^A f(C,t)dt)}{\partial A} = -f(C,A)$

and also $\frac{\partial \int_A^B f(C,t)dt}{\partial B} = f(C,B)$.

The term $\frac{\partial \int_A^B f(C,t)dt}{\partial C} = \frac{\partial \int_A^B f(x,t)dt}{\partial x}$

Conclusion: The formula $\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = \frac{\partial(-\int_B^A f(C,t)dt)}{\partial A}\cdot\frac{da(x)}{dx} + \frac{\partial \int_A^B f(C,t)dt}{\partial B}\cdot\frac{dB}{dx} + \frac{\partial \int_A^B f(C,t)dt}{\partial C}\cdot\frac{dx}{dx}$

Becomes

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = -f(x,a(x))\cdot a'(x) + f(x,b(x))\cdot b'(x) + \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x}dt \cdot 1$$

as we were required to show.

Summary on Chain Rule

If f is a function of n variables, $x_1, x_2, \cdots, x_n$ and each of these variables depends on x. Then f is a function of x only. The Chain Rule then says

$$\frac{df}{dx} = f_1 \cdot \frac{dx_1}{dx} + f_2 \cdot \frac{dx_2}{dx} + \cdots + f_n \cdot \frac{dx_n}{dx}$$

where $f_1 = \frac{\partial f}{\partial x_1}, \cdots, f_n = \frac{\partial f}{\partial x_n}$

Remark:

- We write $\frac{df}{dx}$ because there is only one variable to differentiate (f is ultimately a function of x only).
- Similarly, $\frac{dx_1}{dx}, \cdots, \frac{dx_n}{dx}$ because they depend on one variable

On the other hand, if f is a function of n variables, $x_1, x_2, \cdots, x_n$ and each of these variables depends on more than 1 variable, say u, v. Then f is a function of u and v only. The Chain Rule then says

$$\frac{\partial f}{\partial u} = f_1 \cdot \frac{\partial x_1}{\partial u} + f_2 \cdot \frac{\partial x_2}{\partial u} + \cdots + f_n \cdot \frac{\partial x_n}{\partial u}$$

and

$$\frac{\partial f}{\partial v} = f_1 \cdot \frac{\partial x_1}{\partial v} + f_2 \cdot \frac{\partial x_2}{\partial v} + \cdots + f_n \cdot \frac{\partial x_n}{\partial v}$$