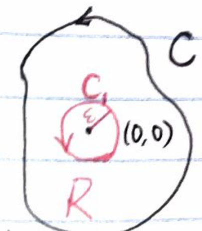
Tutorial 11

1. Let C be any arbitrary smooth, simple closed curve in the plane such that $(0,0)$ lies inside C. Evaluate the following integral.

(a) $\oint_C \frac{2x}{x^2+y^2}dx + \frac{2y}{x^2+y^2}dy$ (b) $\oint_C \frac{2x}{x^2+y^2}dy - \frac{2y}{x^2+y^2}dx$



(a)

Ans: Let C_1 be the circle centered at $(0,0)$ with radius ε small enough such that C_1 lies inside C, and let R be the region enclosed by C and C_1. Then, by Green's theorem, $\oint_C \frac{2x}{x^2+y^2}dx+\frac{2y}{x^2+y^2}dy - \oint_{C_1} \frac{2x}{x^2+y^2}dx+\frac{2y}{x^2+y^2}dy$

$$= \iint_R \left[-\frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2}\right)+\frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2}\right)\right]dx\,dy = 0$$

$$\Rightarrow \oint_C \frac{2x}{x^2+y^2}dx+\frac{2y}{x^2+y^2}dy = \oint_{C_1}\frac{2x}{x^2+y^2}dx+\frac{2y}{x^2+y^2}dy = \int_0^{2\pi}\frac{2\varepsilon\cos\theta}{\varepsilon^2}d(\varepsilon\cos\theta)+\frac{2\varepsilon\sin\theta}{\varepsilon^2}d(\varepsilon\sin\theta)$$

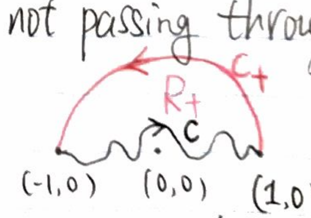
$$= 2\int_0^{2\pi}\cos\theta\, d\cos\theta + \sin\theta\, d\sin\theta = 2\left[\frac{1}{2}\cos^2\theta+\frac{1}{2}\sin^2\theta\right]_0^{2\pi} = 0$$

(b) $\oint_C \frac{2x}{x^2+y^2}dy-\frac{2y}{x^2+y^2}dx - \oint_{C_1}\frac{2x}{x^2+y^2}dy-\frac{2y}{x^2+y^2}dx = \iint_R\left[\frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right)+\frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right)\right]dx\,dy = 0$

$$\Rightarrow \oint_C \frac{2x}{x^2+y^2}dy-\frac{2y}{x^2+y^2}dx = \oint_{C_1}\frac{2x}{x^2+y^2}dy-\frac{2y}{x^2+y^2}dx = \int_0^{2\pi}\frac{2\varepsilon\cos\theta}{\varepsilon^2}d(\varepsilon\sin\theta)+\frac{-2\varepsilon\sin\theta}{\varepsilon^2}d(\varepsilon\cos\theta)$$

$$= 2\int_0^{2\pi}(\cos^2\theta+\sin^2\theta)d\theta = 4\pi$$

2. For any smooth curve C connecting $(-1,0)$ and $(1,0)$ and not passing through $(0,0)$ evaluate $\int_C \frac{2x}{x^2+y^2}dy - \frac{2y}{x^2+y^2}dx$.



Ans: Case 1: the curve C lies above $(0,0)$. Let C_+ be the unit half circle lies above the x-axis and R_+ be the region enclosed by C and C_+. Then, by Green's theorem $\oint_{C+C_+}\frac{2x}{x^2+y^2}dy-\frac{2y}{x^2+y^2}dx = \iint_{R_+}\left[\frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right)\right]dx\,dy = 0$

$$\Rightarrow \int_C \frac{2x}{x^2+y^2}dy-\frac{2y}{x^2+y^2}dx = -\int_{C_+}\frac{2x}{x^2+y^2}dy-\frac{2y}{x^2+y^2}dx = -\int_0^{\pi}2\cos\theta\, d\sin\theta - 2\sin\theta\, d\cos\theta = -2\pi$$

Case 2: the curve C lies below $(0,0)$, by the same argument as case 1,

$$\int_C \frac{2x}{x^2+y^2}dy-\frac{2y}{x^2+y^2}dx = -\int_{2\pi}^{\pi}2\cos\theta\, d\sin\theta - 2\sin\theta\, d\cos\theta = 2\pi$$