Tutorial 12

Exercises 16.6.

19. Let the parametrization be $\vec{r}(x,y) = x\vec{i} + y\vec{j} + (4-y^2)\vec{k}$, $0 \le x \le 1$, $-2 \le y \le 2$ (since $z=0 \Rightarrow 4-y^2=0 \Rightarrow y=\pm 2$) $\Rightarrow \vec{r}_x = \vec{i}$, $\vec{r}_y = \vec{j} - 2y\vec{k}$

$$\Rightarrow \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2y \end{vmatrix} = 2y\vec{j} + \vec{k} \Rightarrow \vec{F}\cdot\vec{n}\,d\sigma = \vec{F}\cdot\frac{\vec{r}_x\times\vec{r}_y}{|\vec{r}_x\times\vec{r}_y|}\cdot|\vec{r}_x\times\vec{r}_y|\,dx\,dy$$

$$= (2xy-3z)\,dx\,dy = [2xy-3(4-y^2)]\,dx\,dy$$

$$\Rightarrow \iint_S \vec{F}\cdot\vec{n}\,d\sigma = \int_0^1\int_{-2}^2 (2xy-12+3y^2)\,dy\,dx = \int_0^1 \left[xy^2-12y+y^3\right]_{-2}^2 dx$$

$$= \int_0^1 -32\,dx = -32$$

Exercises 16.7.

8. $$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z+\frac{1}{2+x} & \tan^{-1}y & x+\frac{1}{4+z} \end{vmatrix} = -2\vec{j}.\quad f(x,y,z) = 4x^2+y+z^2-4 \Rightarrow \nabla f = 8x\vec{i}+\vec{j}+2z\vec{k}$$

$$\Rightarrow \vec{n} = \frac{\nabla f}{|\nabla f|}$$

Let $y=0$ in $4x^2+y^2+z^2=4$, then $4x^2+z^2=4 \Leftrightarrow x^2+\left(\frac{z}{2}\right)^2=1$, an ellipse.

Take $\vec{p}=\vec{j}$, then $|\nabla f\cdot p| = 1$ and $d\sigma = \frac{|\nabla f|}{|\nabla f\cdot p|}\,dx\,dz = |\nabla f|\,dx\,dz$

$$\Rightarrow \iint_S \nabla\times\vec{F}\cdot\vec{n}\,d\sigma = \iint_R (\nabla\times\vec{F})\cdot\nabla f\,dx\,dz = \iint_R -2\,dx\,dz = -2\,\text{Area}(R)$$

$= -2\cdot(\pi\cdot 1\cdot 2) = -4\pi$ where R is the elliptic region in the xz-plane enclosed by $x^2+(z/2)^2=1$.

Exercises 16.8

4. Let $\rho = \sqrt{x^2+y^2+z^2}$, $\vec{F} = \frac{x}{\rho}\vec{i} + \frac{y}{\rho}\vec{j} + \frac{z}{\rho}\vec{j} \Rightarrow \frac{\partial}{\partial x}\left(\frac{x}{\rho}\right) = \frac{1}{\rho} - \frac{x^2}{\rho^3}$, $\frac{\partial}{\partial y}\left(\frac{y}{\rho}\right) = \frac{1}{\rho} - \frac{y^2}{\rho^3}$, $\frac{\partial}{\partial z}\left(\frac{z}{\rho}\right) = \frac{1}{\rho} - \frac{z^2}{\rho^3} \Rightarrow \text{div}\vec{F} = \frac{2}{\rho} \Rightarrow \iint_{\partial D} \vec{F}\cdot\vec{n}\,ds = \iiint_D \text{div}\vec{F}\,dV = ~~\int_1^2\int_0^\pi\int_0^{2\pi}\frac{2}{\rho}~~$

$$= \int_1^2\int_0^\pi\int_0^{2\pi}\frac{2}{\rho}\cdot(\rho^2\sin\phi)\,d\theta\,d\phi\,d\rho = 4\pi\int_1^2\int_0^\pi \rho\sin\phi\,d\phi\,d\rho$$

$$= 8\pi\int_1^2 \rho\,d\rho = 12\pi$$