Solutions to HW9

§16.5

17 Parametrize the surface by $(\rho,\theta) \mapsto \vec{r} = (\rho\cos\theta, \rho\sin\theta, 1-\frac{1}{2}\rho\sin\theta)$ $\quad \rho \in [0,1]$, $\theta \in [0,2\pi]$

$$\partial_\rho \vec{r} = (\cos\theta, \sin\theta, -\tfrac{1}{2}\sin\theta)$$

$$\partial_\theta \vec{r} = (-\rho\sin\theta, \rho\cos\theta, -\tfrac{1}{2}\rho\cos\theta)$$

$$\partial_\rho \vec{r} \times \partial_\theta \vec{r} = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & -\frac{1}{2}\sin\theta \\ -\rho\sin\theta & \rho\cos\theta & -\frac{1}{2}\rho\cos\theta \end{vmatrix} = (0, \tfrac{\rho}{2}, \rho)$$

$$|\partial_\rho \vec{r} \times \partial_\theta \vec{r}| = \sqrt{0^2 + (\tfrac{\rho}{2})^2 + \rho^2} = \frac{\sqrt{5}}{2}\rho$$

$$\Rightarrow \text{surface area} = \int_0^{2\pi}\int_0^1 \frac{\sqrt{5}}{2}\rho \, d\rho\, d\theta = \frac{\sqrt{5}}{2} \times 2\pi \times \frac{1}{2} = \frac{\sqrt{5}\pi}{2} \,//$$

20 Parametrize the surface by $(\rho,\theta) \mapsto \vec{r} = (\rho\cos\theta, \rho\sin\theta, \frac{\rho}{3})$ $\quad \rho \in [3,4]$, $\theta \in [0,2\pi]$

$$\partial_\rho \vec{r} = (\cos\theta, \sin\theta, \tfrac{1}{3})$$

$$\partial_\theta \vec{r} = (-\rho\sin\theta, \rho\cos\theta, 0)$$

$$\partial_\rho \vec{r} \times \partial_\theta \vec{r} = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & \frac{1}{3} \\ -\rho\sin\theta & \rho\cos\theta & 0 \end{vmatrix} = (-\tfrac{\rho}{3}\cos\theta, -\tfrac{\rho}{3}\sin\theta, \rho)$$

$$|\partial_\rho \vec{r} \times \partial_\theta \vec{r}| = \sqrt{(-\tfrac{\rho}{3}\cos\theta)^2 + (-\tfrac{\rho}{3}\sin\theta)^2 + \rho^2} = \frac{\sqrt{10}}{3}\rho$$

$$\Rightarrow \text{surface area} = \int_0^{2\pi}\int_3^4 \frac{\sqrt{10}}{3}\rho \, d\rho\, d\theta = \frac{\sqrt{10}}{3} \times 2\pi \times \frac{16-9}{2} = \frac{7\sqrt{10}\pi}{3} \,/\!/$$

24 Parametrize the surface by $(\rho,\theta) \mapsto \vec{r} = (\rho\cos\theta, \rho\sin\theta, \rho^2)$ $\rho \in [1,2]$, $\theta \in [0,2\pi]$

$$\partial_\rho \vec{r} \times \partial_\theta \vec{r} = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 2\rho \\ -\rho\sin\theta & \rho\cos\theta & 0 \end{vmatrix} = (-2\rho^2\cos\theta, -2\rho^2\sin\theta, \rho)$$

$$|\partial_\rho \vec{r} \times \partial_\theta \vec{r}| = \rho\sqrt{1+4\rho^2}$$

$$\Rightarrow \text{surface area} = \int_0^{2\pi}\int_1^2 \rho\sqrt{1+4\rho^2}\, d\rho\, d\theta$$

$$= 2\pi \left[\frac{1}{12}\sqrt{1+4\rho^2}^{\,3}\right]_1^2 = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5}) \quad //$$

26 Parametrize the surface by $(\theta,\phi) \mapsto \vec{r} = (2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi)$

$\theta \in [0,2\pi]$, $\phi \in [\frac{\pi}{6}, \frac{2\pi}{3}]$

$$\partial_\theta \vec{r} \times \partial_\phi \vec{r} = \begin{vmatrix} i & j & k \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \end{vmatrix}$$

$$= (-4\sin^2\phi\cos\theta, -4\sin^2\phi\sin\theta, -4\sin\phi\cos\phi)$$

$$|\partial_\theta \vec{r} \times \partial_\phi \vec{r}| = 4\sin\phi$$

$$\Rightarrow \text{Surface area} = \int_{\frac{\pi}{6}}^{\frac{2\pi}{3}}\int_0^{2\pi} 4\sin\phi\, d\theta\, d\phi$$

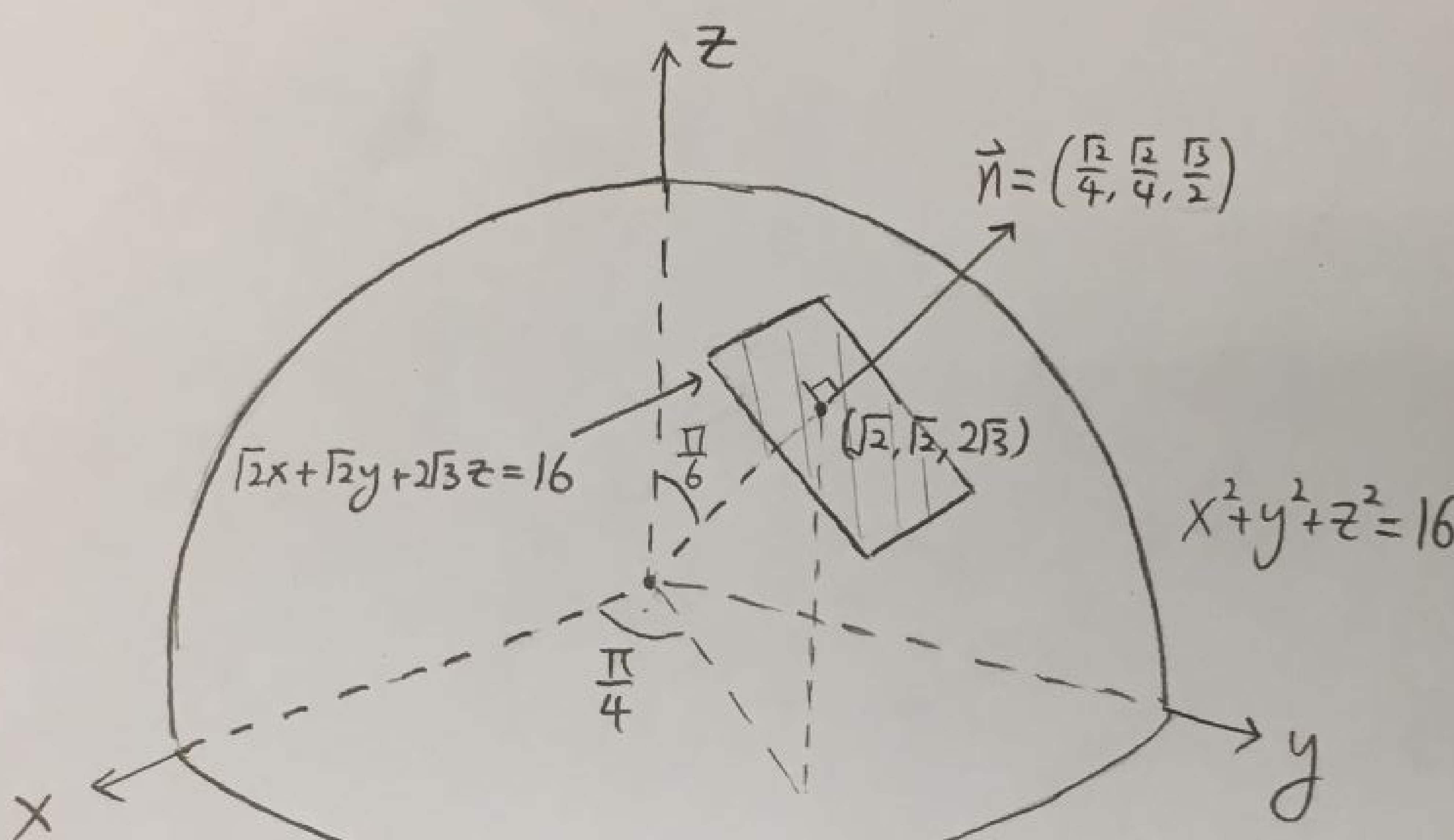
$$= 4 \times 2\pi \times \left[-\cos\phi\right]_{\frac{\pi}{6}}^{\frac{2\pi}{3}} = 4\pi(1+\sqrt{3}) \quad //$$

28 Cartesian eqn $f = x^2+y^2+z^2=16$, $z \geq 0$

$\nabla f = (2x, 2y, 2z)$

$\Rightarrow$ normal plane: $(x-\sqrt{2})(2\sqrt{2})+(y-\sqrt{2})(2\sqrt{2})+(z-2\sqrt{3})(4\sqrt{3})=0$

$\Leftrightarrow \sqrt{2}x+\sqrt{2}y+2\sqrt{3}z-16=0$ //



32b) By a), $\vec{r}(u,v) = f(u)\vec{i} + (g(u)\cos v)\vec{j} + (g(u)\sin v)\vec{k}$

Let $f(u)=u^2$, $g(u)=u$, $u\in\mathbb{R}_{\geq 0}$

$\Rightarrow \vec{r}(u,v)=(u^2, u\cos v, u\sin v)$ $\quad u\in\mathbb{R}_{\geq 0}, v\in[0,2\pi]$ //

38 Project the surface into xy plane, we get $R=\{2\leq x^2+y^2\leq 6\}$, $\vec{p}=\vec{k}$

$\Rightarrow$ surface area $= \iint_R \frac{|\nabla f|}{|\nabla f\cdot\vec{k}|}\,dA = \int_0^{2\pi}\int_{\sqrt{2}}^{\sqrt{6}} \frac{\sqrt{(2x)^2+(2y)^2+(-1)^2}}{\sqrt{0^2+0^2+(-1)^2}}\,r\,dr\,d\theta$

$= 2\pi\int_{\sqrt{2}}^{\sqrt{6}} r\sqrt{1+4r^2}\,dr = 2\pi\left[\frac{1}{12}\sqrt{1+4r^2}^{\,3}\right]_{\sqrt{2}}^{\sqrt{6}} = \frac{49\pi}{3}$ //

42 Project the surface into xy plane, we get $R=\{x^2+y^2\le 1\}$, $\vec{p}=\vec{k}$

$$\Rightarrow \text{surface area} = \iint_R \frac{\sqrt{(2x)^2+(2y)^2+(2z)^2}}{\sqrt{0^2+0^2+(2z)^2}}\, dA$$

$$= \int_0^{2\pi}\int_0^1 \frac{2\sqrt{2}}{2z}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{\sqrt{2}r}{\sqrt{2-r^2}}\, dr\, d\theta$$

$$= 2\sqrt{2}\pi\left[-\sqrt{2-r^2}\right]_0^1$$

$$= 2\pi(2-\sqrt{2}) \;//$$

55 $\partial_x\vec{r} = (1, \partial_x f, 0)$, $\partial_z\vec{r} = (0, \partial_z f, 1)$

$$\partial_x\vec{r}\times\partial_z\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & \partial_x f & 0 \\ 0 & \partial_z f & 1 \end{vmatrix} = (\partial_x f, -1, \partial_z f)$$

$$\Rightarrow d\sigma = |\partial_x\vec{r}\times\partial_z\vec{r}|\, dx\,dz = \sqrt{1+(\partial_x f)^2+(\partial_z f)^2}\, dx\,dz \;//$$

56 b) By a), we consider $\vec{r}(x,\theta) = (x, f(x)\cos\theta, f(x)\sin\theta)$ $x\in[a,b]$, $\theta\in[0,2\pi]$

$\partial_x\vec{r} = (1, f'(x)\cos\theta, f'(x)\sin\theta)$, $\partial_\theta\vec{r} = (0, -f(x)\sin\theta, f(x)\cos\theta)$

$$\partial_x\vec{r}\times\partial_\theta\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f'\cos\theta & f'\sin\theta \\ 0 & -f\sin\theta & f\cos\theta \end{vmatrix} = (ff', -f\cos\theta, -f\sin\theta)$$

$$|\partial_x\vec{r}\times\partial_\theta\vec{r}| = \sqrt{(ff')^2+(-f\cos\theta)^2+(-f\sin\theta)^2} = f\sqrt{1+(f')^2}$$

$$\Rightarrow \text{surface area } A = \int_0^{2\pi}\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx\,d\theta = 2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx \;//$$