Solutions to HW8

§16.3

9. $f = xe^{y+2z}$ // $\left(\text{check: } \frac{\partial f}{\partial x} = e^{y+2z}, \frac{\partial f}{\partial y} = xe^{y+2z}, \frac{\partial f}{\partial z} = 2xe^{y+2z}\right)$

11. $f = x\ln x - x + \tan(x+y) + \frac{1}{2}\ln(y^2+z^2)$ //

$\left(\text{Check: } \frac{\partial f}{\partial x} = \ln x + \sec^2(x+y), \frac{\partial f}{\partial y} = \sec^2(x+y) + \frac{y}{y^2+z^2}, \frac{\partial f}{\partial z} = \frac{z}{y^2+z^2}\right)$

22. a potential f for the field $F = \left(\frac{2x}{x^2+y^2+z^2}, \frac{2y}{x^2+y^2+z^2}, \frac{2z}{x^2+y^2+z^2}\right)$ is $f = \ln(x^2+y^2+z^2)$

So $\int_{(-1,-1,-1)}^{(2,2,2)} F\cdot(dx,dy,dz) = \int_{(-1,-1,-1)}^{(2,2,2)} df = f(2,2,2) - f(-1,-1,-1)$

$= \ln(12) - \ln(3) = 2\ln 2$ //

26. Notice that the field $F = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right)$ has a potential function

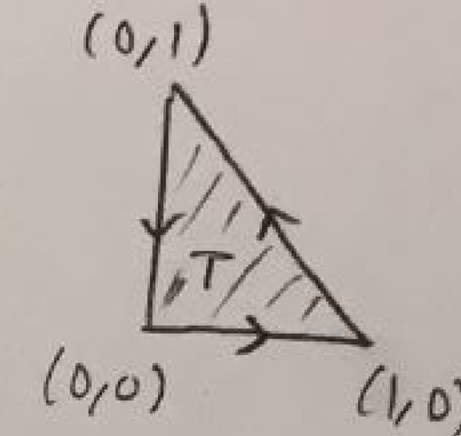
$f = \sqrt{x^2+y^2+z^2} \Rightarrow$ result.

28. A potential function for F is $f = e^x \ln y + y\sin z$

$\left(\text{check: } \frac{\partial f}{\partial x} = e^x\ln y, \frac{\partial f}{\partial y} = \frac{e^x}{y} + \sin z, \frac{\partial f}{\partial z} = y\cos z\right)$

§16.4

21 $\oint_C (y^2 dx + x^2 dy)$

C: 

$$= \iint_T \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2)\right) dy dx$$

$$= \int_0^1 \int_0^{1-x} 2(x-y)\, dy dx$$

$$= \int_0^1 2x(1-x) - (1-x)^2 dx$$

$$= \frac{2}{6} - \frac{1}{3} = 0 \,//$$ (or use substitution $x \leftrightarrow y$)

25 Area of R bounded by $r(t) = (a\cos t, a\sin t)$ $t \in [0, 2\pi]$

$$= \oint_C \frac{1}{2} x\, dy - \frac{1}{2} y\, dx$$

$$= \int_0^{2\pi} \left[\frac{1}{2} \cdot a\cos t(a\cos t) - \frac{1}{2} a\sin t(-a\sin t)\right] dt$$

$$= \frac{1}{2} a^2 \int_0^{2\pi} dt = \pi a^2 \,//$$

27 Area of R bounded by $r(t) = (\cos^3 t, \sin^3 t)$ $t \in [0, 2\pi]$

$$= \oint \frac{1}{2} x\, dy - \frac{1}{2} y\, dx$$

$$= \int_0^{2\pi} \left[\frac{1}{2} \cos^3 t (3\sin^2 t \cos t) - \frac{1}{2} \sin^3 t (-3\cos^2 t \sin t)\right] dt$$

$$= \frac{3}{2} \int_0^{2\pi} \cos^2 t \sin^2 t\, dt = \frac{3\pi}{8} \,//$$

33 Recall Green's theorem $\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dy\,dx$

- Let $(M,N) = (x,0) \Rightarrow \oint_C x\,dy = \iint_R 1 \cdot dy\,dx = \text{Area}(R)$
- Let $(M,N) = (0,y) \Rightarrow -\oint_C y\,dx = \iint_R 1 \cdot dy\,dx = \text{Area}(R)$ //

39 a) $\nabla f = \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right)$, C parametrized by $(a\cos t, a\sin t)$, $t \in [0, 2\pi]$

$$\oint_C \nabla f \cdot n\,ds = \oint_C \frac{2x}{x^2+y^2}\,dy - \frac{2y}{x^2+y^2}\,dx$$

$$= \int_0^{2\pi} [2\cos t(\cos t) - 2\sin t(-\sin t)]\,dt$$

$$= 4\pi \;//$$

b) Let K be a smooth simple closed curve $\subset \mathbb{R}^2$ w/ $0 \notin C$.

Then K bounds a region R.

Case 1 $\quad 0 \in R$: Draw a small circle $C_\varepsilon = \{x^2+y^2 = \varepsilon^2\}$ (oriented counterclockwise) st $C_\varepsilon \cap K = \emptyset$

Let $R' = R - \{x^2+y^2 \le \varepsilon^2\}$.

Then Green's thm $\Rightarrow$

$$\left[\begin{aligned} &\nabla \cdot \nabla f \\ &= \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right) \\ &= \frac{2(x^2+y^2) - 4x^2}{(x^2+y^2)^2} + \frac{2(x^2+y^2) - 4y^2}{(x^2+y^2)^2} \\ &= 0 \end{aligned}\right] \longrightarrow \left(\oint_K + \oint_{-C_\varepsilon}\right) \nabla f \cdot \vec{n}\,ds = \iint_{R'} \nabla \cdot \nabla f\,dA = 0$$

$$\Rightarrow \oint_K \nabla f \cdot \vec{n}\,ds = -\oint_{-C_\varepsilon} \nabla f \cdot \vec{n}\,ds = \oint_{C_\varepsilon} \nabla f \cdot \vec{n}\,ds = 4\pi \;//$$

Case 2 $\quad 0 \notin R$: Then Green's thm in original form $\Rightarrow \oint_K \nabla f \cdot \vec{n}\,ds = 0$ //