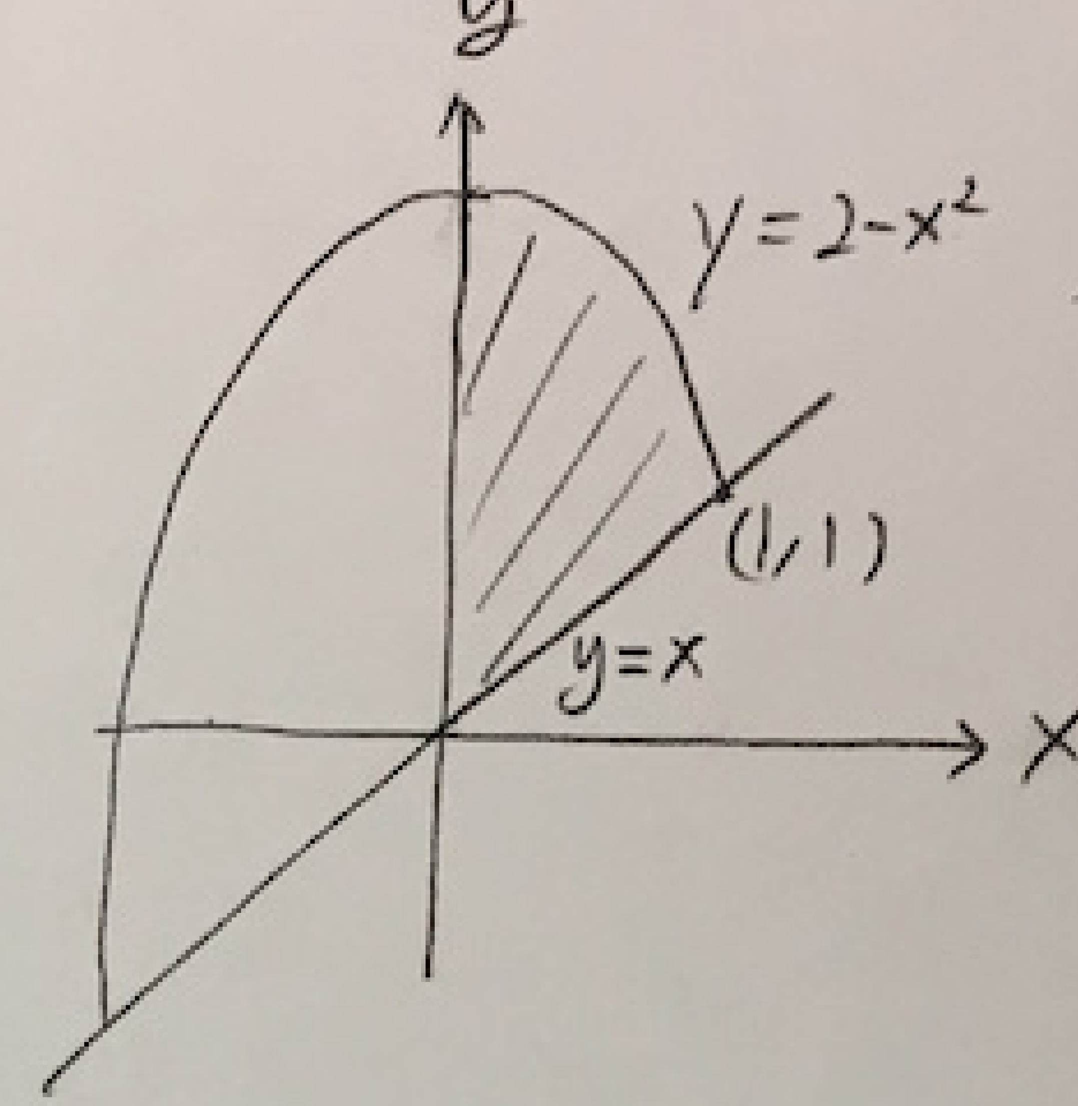
Solutions to HW5

§15.6



1 $m = \int_0^1 \int_x^{2-x^2} 3 \, dy\, dx$

$= 3 \int_0^1 [(2-x^2) - x] \, dx$

$= 3 \left(2 - \frac{1}{3} - \frac{1}{2}\right)$

$= \frac{7}{2}$

$M_x = \int_0^1 \int_x^{2-x^2} y \cdot 3 \cdot dy\, dx$

$= \frac{3}{2} \int_0^1 [(2-x^2)^2 - x^2] \, dx$

$= \frac{3}{2} \int_0^1 (4 - 4x^2 + x^4 - x^2) \, dx$

$= \frac{3}{2} \left(4 - \frac{5}{3} + \frac{1}{5}\right)$

$= \frac{38}{10}$

$M_y = \int_0^1 \int_x^{2-x^2} x \cdot 3 \cdot dy\, dx$

$= 3 \int_0^1 x (2 - x^2 - x) \, dx$

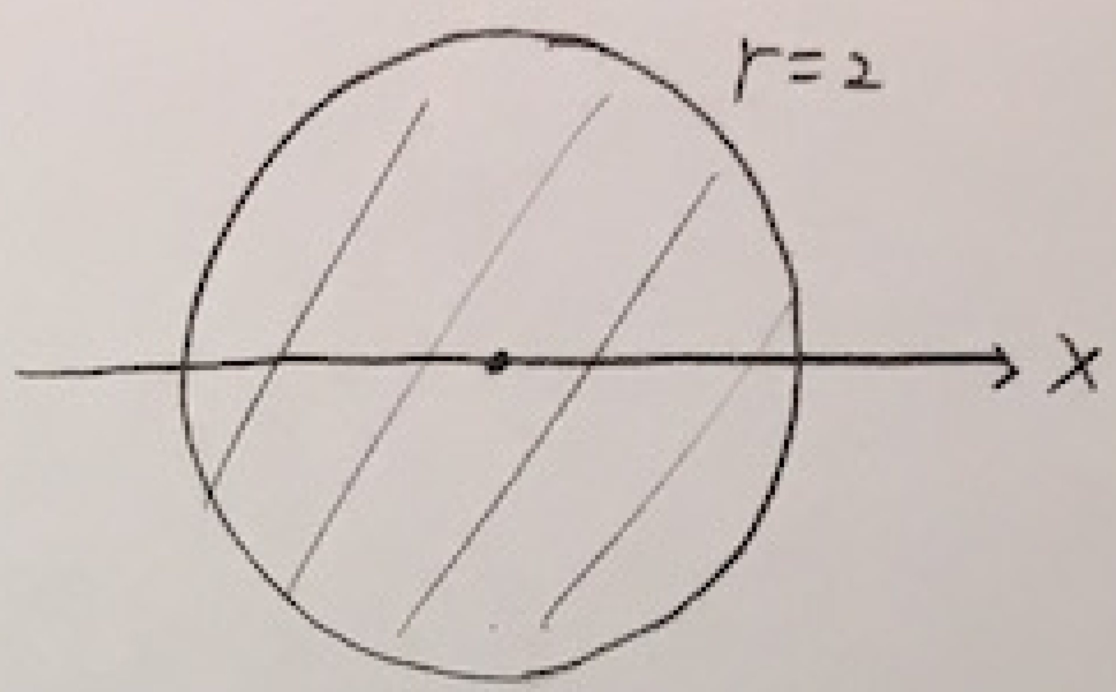
$= 3 \left(2 \times \frac{1}{2} - \frac{1}{4} - \frac{1}{3}\right)$

$= \frac{5}{4}$

$\therefore (\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{\frac{5}{4}}{\frac{7}{2}}, \frac{\frac{38}{10}}{\frac{7}{2}}\right)$

$= \left(\frac{5}{14}, \frac{38}{35}\right)$ //

7 $I_x = \int_0^{2\pi}\int_0^2 y^2 \cdot 1 \cdot r\,dr\,d\theta$



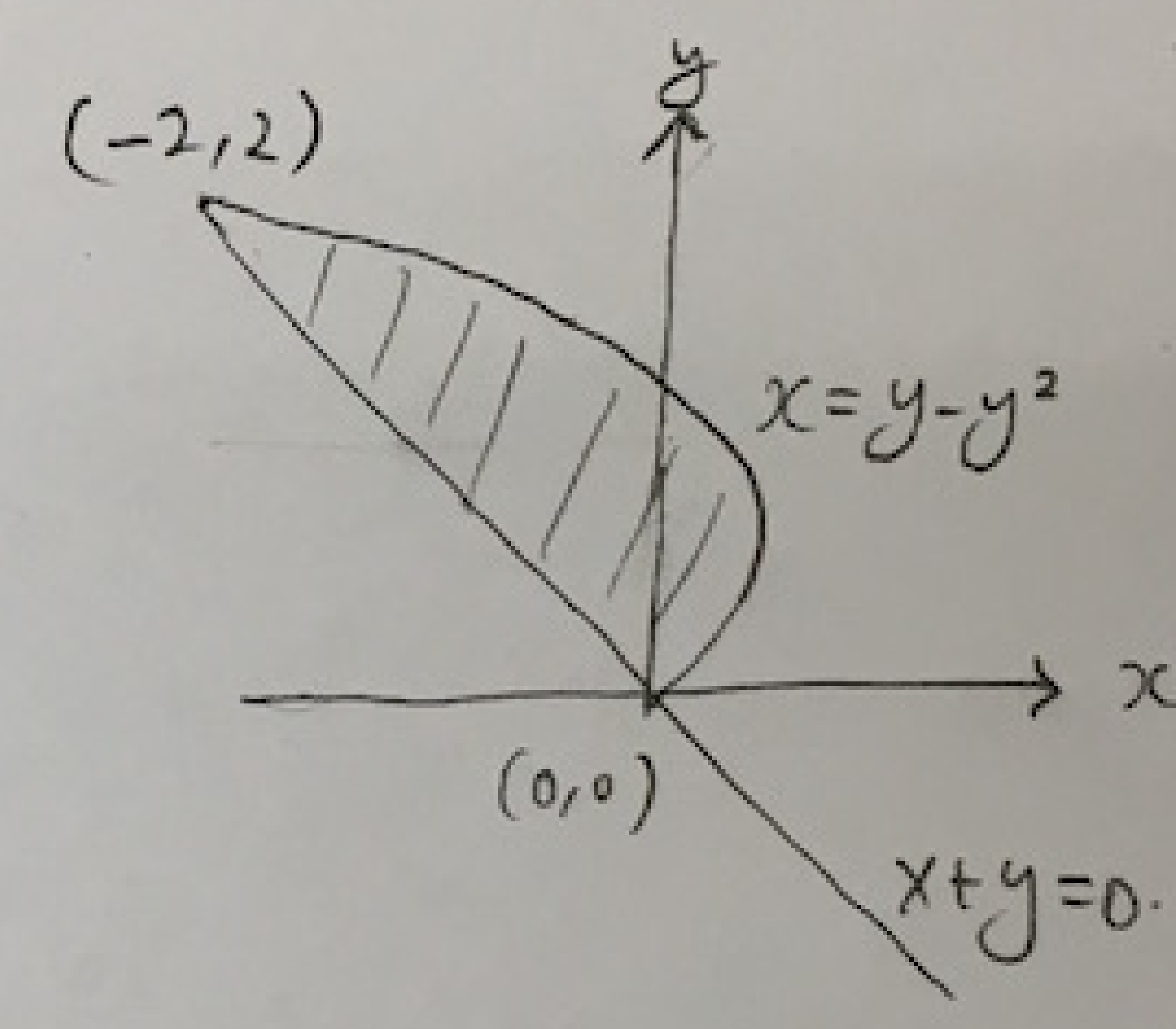
$= \int_0^{2\pi}\int_0^2 r^2 \sin^2\theta \cdot r\,dr\,d\theta$

$= \left(\int_0^{2\pi} \sin^2\theta\right)\left(\int_0^2 r^3\,dr\right)$

$= 4\pi$ //

By symmetry, $I_y = I_x = 4\pi$ //, and hence $I_0 = I_x + I_y = 8\pi$ //

11 $I_x = \int_0^2 \int_{-y}^{y-y^2} y^2 \cdot (x+y)\,dx\,dy$



$= \int_0^2 \left[\frac{x^2y^2}{2} + xy^3\right]_{-y}^{y-y^2} dy$

$= \int_0^2 \left[\frac{(y-y^2)^2 - y^2}{2} \times y^2 + (y-y^2+y)y^3\right] dy$

$= \int_0^2 \left(-y^5 + \frac{y^6}{2} + 2y^4 - y^5\right) dy$

$= 32\left(-\frac{2}{6} + \frac{1}{2\times 7} \times 2^2 + \frac{2}{5} - \frac{2}{6}\right)$

$= \frac{64}{105}$ //

13

$$M_x = \int_0^1 \int_x^{2-x} y\,(6x+3y+3)\,dy\,dx$$

$$= \int_0^1 \left[3xy^2 + y^3 + \frac{3y^2}{2}\right]_x^{2-x} dx$$

$$= \int_0^1 \left\{\left(3x+\frac{3}{2}\right)\left[(2-x)^2-x^2\right] + (2-x)^3 - x^3\right\} dx$$

$$= \int_0^1 3\left(x+\frac{1}{2}\right)\times 4(1-x)\,dx + \left[-\frac{1}{4}(2-x)^4\right]_0^1 - \frac{1}{4}$$

$$= 12\left(\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}-\frac{1}{3}\right) - \frac{1}{4} + \frac{16}{4} - \frac{1}{4}$$

$$= \frac{17}{2}$$

$$M_y = \int_0^1 \int_x^{2-x} x\,(6x+3y+3)\,dy\,dx$$

$$= \int_0^1 \left[(6x^2+3x)y + \frac{3xy^2}{2}\right]_x^{2-x} dx$$

$$= \int_0^1 \left[(6x^2+3x)(2-2x) + \frac{3x}{2}(4-4x)\right] dx$$

$$= 6\left(\frac{1}{2}+\frac{1}{3}-2\times\frac{1}{4}\right) + 6\left(\frac{1}{2}-\frac{1}{3}\right)$$

$$= 3$$

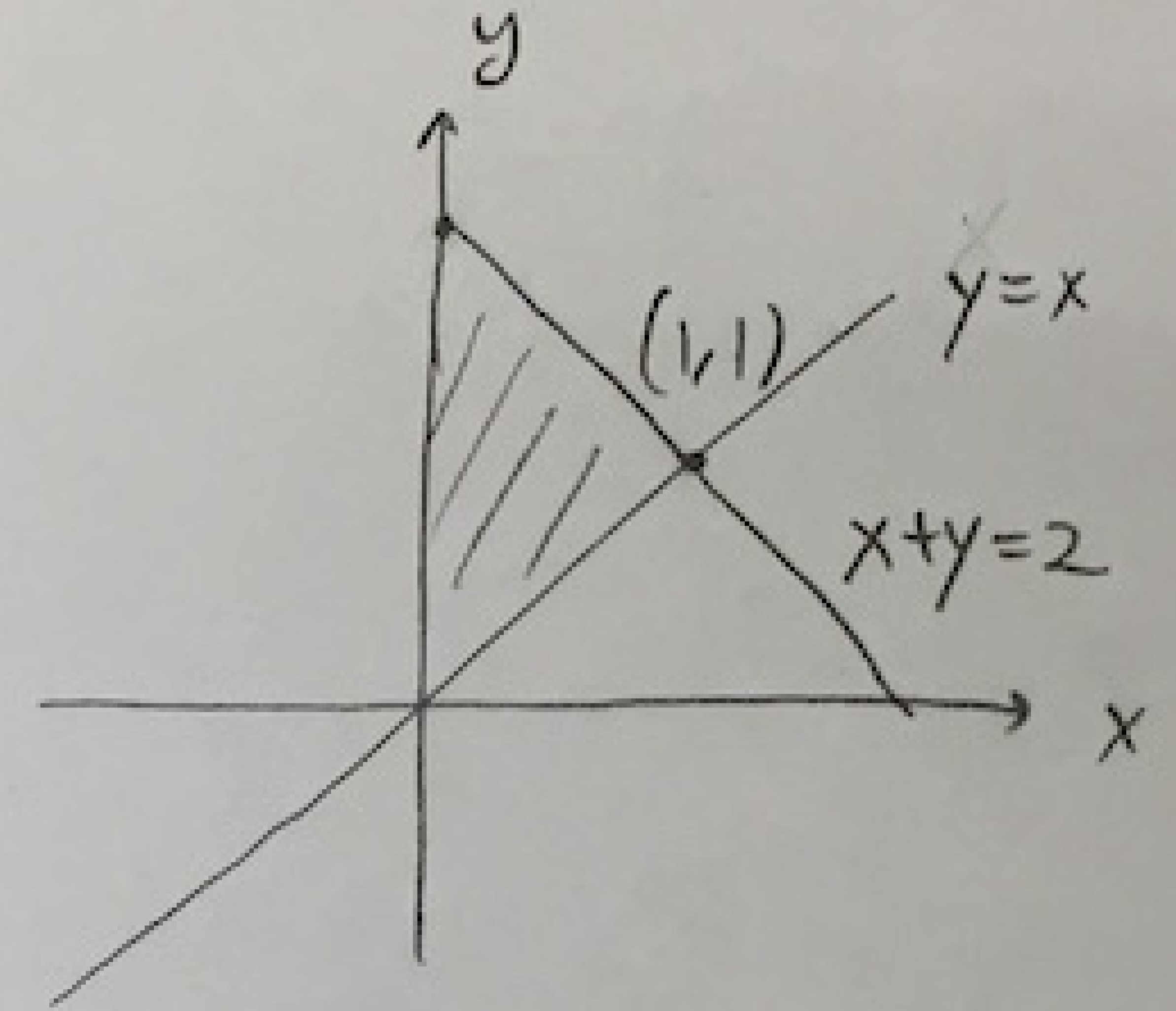
13 (cont)

$$m = \int_0^1 \int_x^{2-x} (6x+3y+3)\, dy\, dx$$

$$= \int_0^1 (6x+3)(2-2x) + \frac{3}{2}(4-4x)\, dx$$

$$= 6\left(1+\frac{1}{2}-2\times\frac{1}{3}\right) + 6\left(1-\frac{1}{2}\right)$$

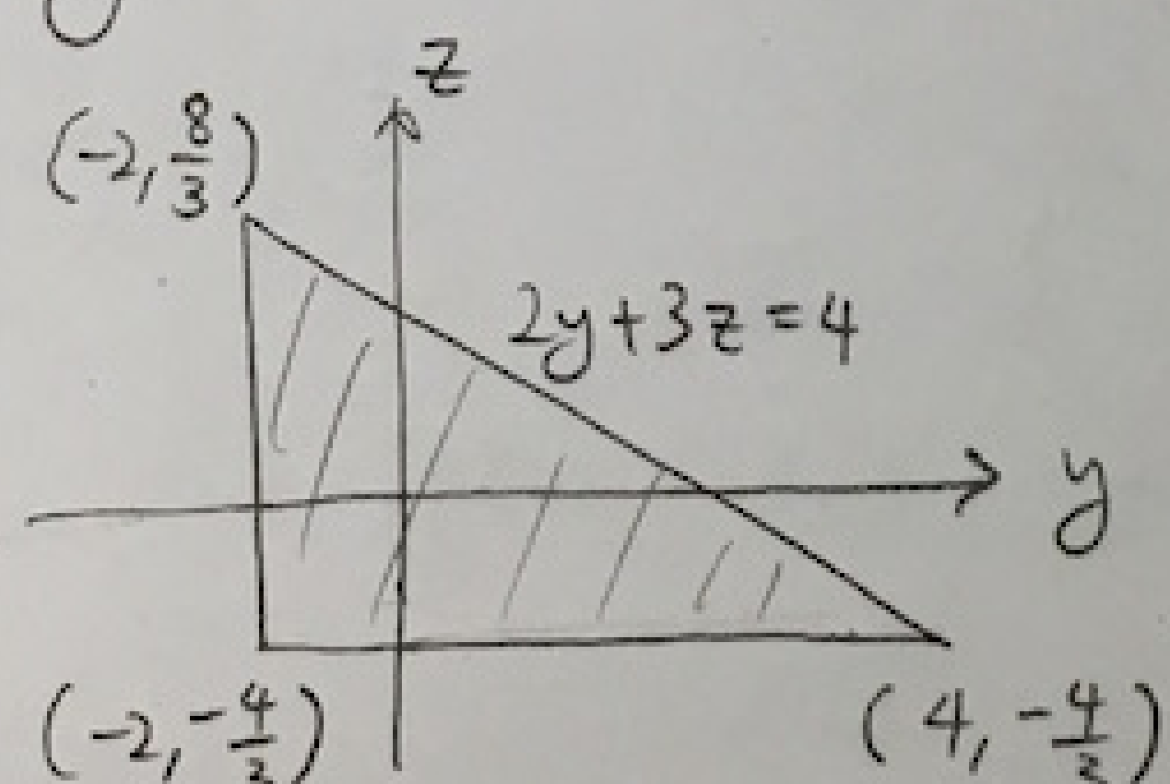
$$= 8$$



$$\therefore (\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

$$= \left(\frac{3}{8}, \frac{17}{16}\right)$$

22 The lateral triangle of the solid in the yz plane is given by



$$J_x \triangleq \int_{-3}^{3} \int_{-2}^{4} \int_{-\frac{4}{3}}^{\frac{4-2y}{3}} x^2\, dz\, dy\, dx \quad (\leftarrow \text{This is } \underline{\text{NOT}}\ I_x)$$

$$= \left[\frac{x^3}{3}\right]_{-3}^{3} \times \text{area of triangle}$$

$$= 18 \times \frac{1}{2} \times 6 \times 4 = 216$$

22 (cont)

$$J_y \triangleq \int_{-3}^{3}\int_{-2}^{4}\int_{-\frac{4}{3}}^{\frac{4-2y}{3}} y^2 \, dz \, dy \, dx$$

$$= 6 \times \int_{-2}^{4} y^2\left(\frac{4-2y}{3} + \frac{4}{3}\right) dy$$

$$= 6 \times \frac{1}{3}\left[\frac{8y^3}{3} - \frac{y^4}{2}\right]_{-2}^{4}$$

$$= 2(192 - 120)$$

$$= 144$$

$$J_z \triangleq \int_{-3}^{3}\int_{-2}^{4}\int_{-\frac{4}{3}}^{\frac{4-2y}{3}} z^2 \, dz \, dy \, dx$$

$$= 6 \times \frac{1}{3}\int_{-2}^{4}\left[\left(\frac{4-2y}{3}\right)^3 + \left(\frac{4}{3}\right)^3\right] dy$$

$$= \frac{2}{27}\left[-\frac{(4-2y)^4}{2 \times 4} + 4^3 \times y\right]_{-2}^{4}$$

$$= 64.$$

density $\rightarrow \delta = \frac{\text{mass}}{\text{volume}} = \frac{m}{\frac{1}{2} \times 6 \times 4 \times 6} = \frac{m}{72}$

$\therefore \; I_x = \frac{m}{72}(J_y + J_z) \; ; \; I_y = \frac{m}{72}(J_x + J_z) \; ; \; I_z = \frac{m}{72}(J_x + J_y)$

$= \frac{26}{9}m$ // $\quad = \frac{35}{9}m$ // $\quad = 5m$ //

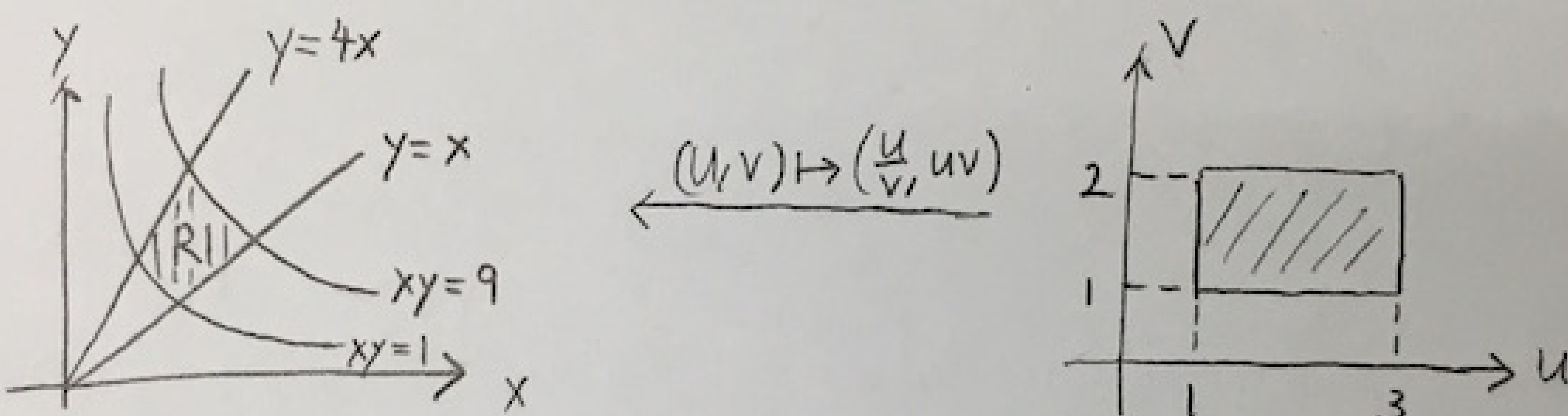
9

$$\begin{cases} x = \frac{u}{v} \\ y = uv \end{cases} \Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{matrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{matrix}\right| = \frac{2u}{v}$$

$$\therefore \iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\,dy = \int_1^3 \int_1^2 (v+u)\left(\frac{2u}{v}\right) dv\,du$$

$$= 2\int_1^3 [u + (\ln 2)u^2]\,du$$

$$= 8 + \frac{52\ln 2}{3}$$

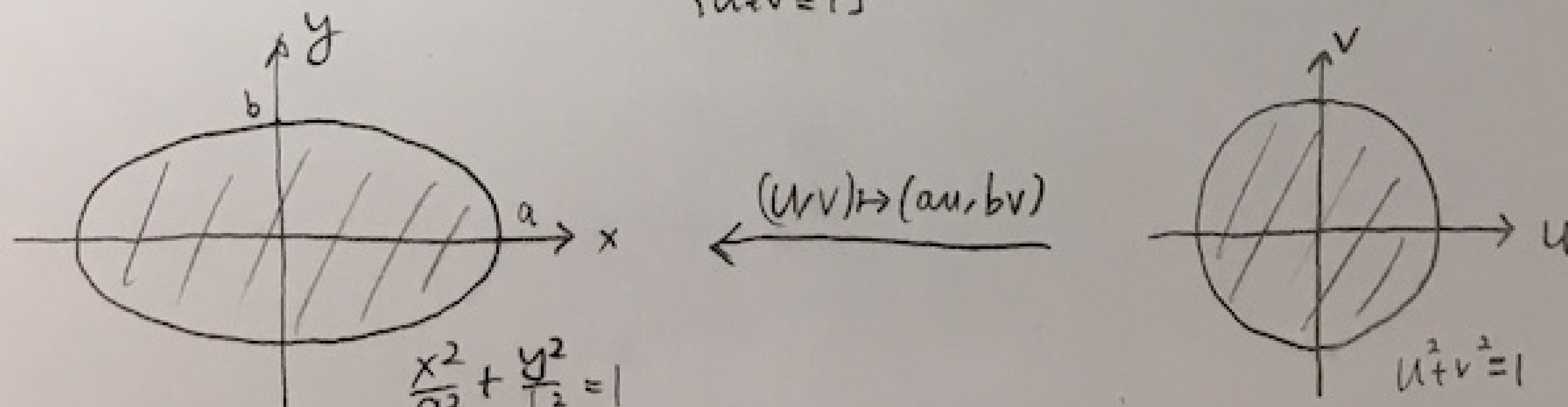


12

$$\begin{cases} x = au \\ y = bv \end{cases} \Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{matrix} a & 0 \\ 0 & b \end{matrix}\right| = ab$$

$$\therefore \text{Area of } \left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\right\} = \iint_{\left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\right\}} dx\,dy$$

$$= \iint_{\{u^2+v^2 \le 1\}} ab\,du\,dv = \pi ab$$



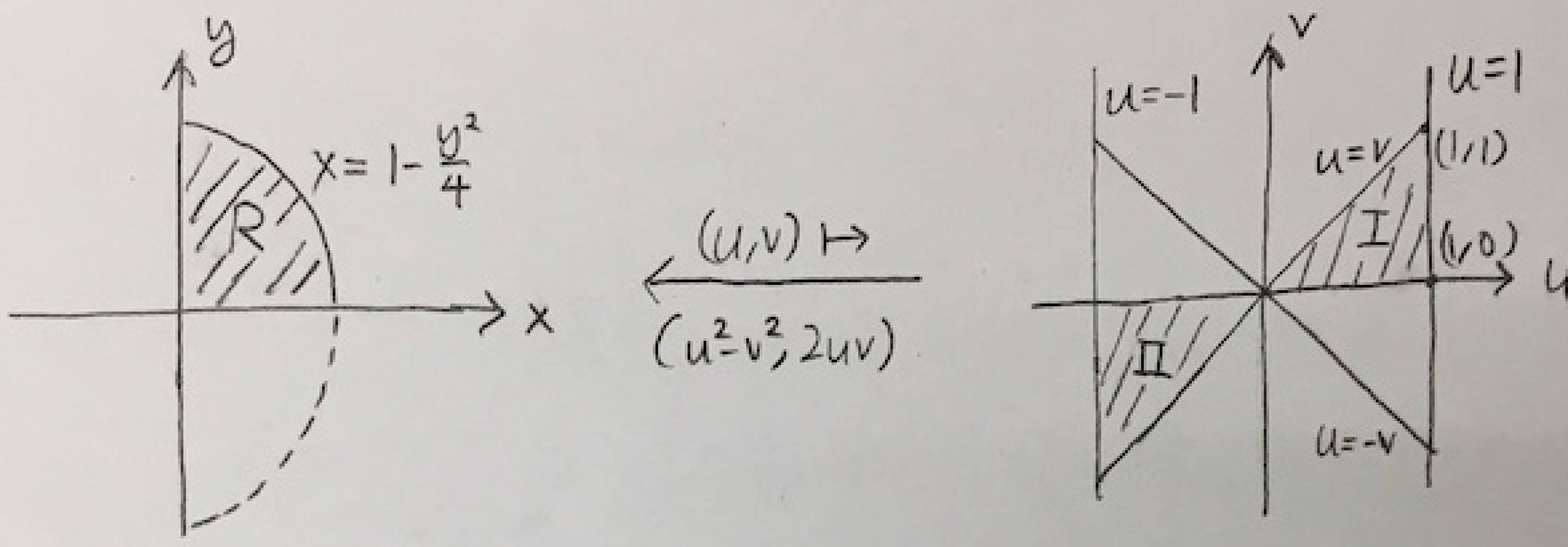
16 $\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases} \Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2+v^2)$

Notice that $x \le 1 - \frac{y^2}{4} \iff u^2 - v^2 \le 1 - u^2v^2 \iff (u^2-1)(v^2+1) \le 0 \iff |u| \le 1$

$x \ge 0 \iff |u| \ge |v|$

$y \ge 0 \iff (u, v \ge 0)$ OR $(u, v \le 0)$

So



Where each of I and II is mapped diffeomorpically onto R.

$$\therefore \int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2+y^2}\, dy\, dx = \int_0^1 \int_0^u \sqrt{(u^2-v^2)^2 + (2uv)^2} \cdot 4(u^2+v^2)\, dv\, du$$

$$= \int_0^1 \int_0^u 4(u^2+v^2)^2\, dv\, du$$

$$= 4 \int_0^1 \left[u^4(u) + \frac{2}{3}u^2(u^3) + \frac{1}{5}u^5\right] du$$

$$= 4\left(1 + \frac{2}{3} + \frac{1}{5}\right) \times \frac{1}{6}$$

$$= \frac{56}{45} \text{ //}$$

23 $\begin{cases} x = au \\ y = bv \\ z = cw \end{cases} \Rightarrow \left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$

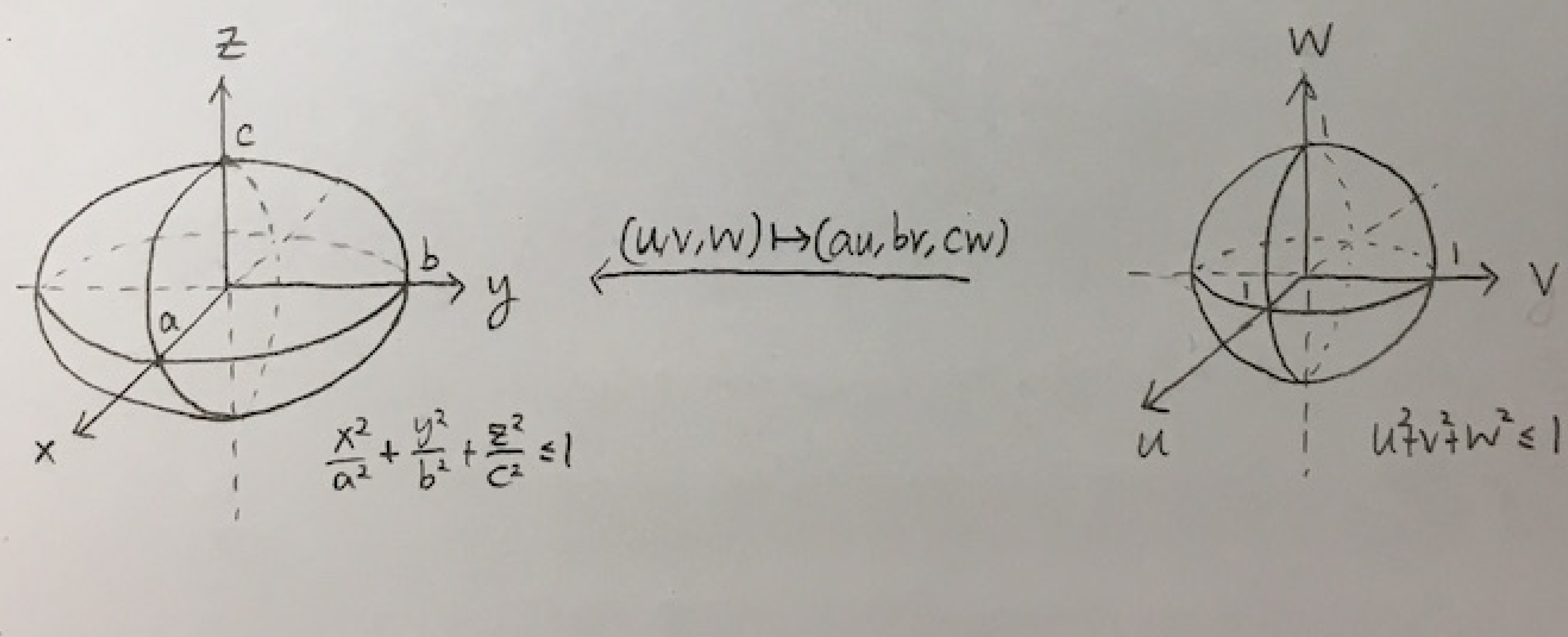
$$\therefore \iiint_{\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\right\}} |xyz|\,dx\,dy\,dz = \iiint_{\{u^2+v^2+w^2\le 1\}} |au\cdot bv\cdot cw|\,abc\,du\,dv\,dw$$

$$= a^2b^2c^2 \times 8 \iiint_{\substack{\{u^2+v^2+w^2\le 1\}\cap \\ \{u,v,w\ge 0\}}} uvw\,du\,dv\,dw$$

$$= 8a^2b^2c^2 \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 8a^2b^2c^2\left(\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\,d\theta\right)\left(\int_0^{\frac{\pi}{2}}\sin^3\phi\cos\phi\,d\phi\right)\left(\int_0^1 \rho^5\,d\rho\right)$$

$$= 8a^2b^2c^2 \times \frac{1}{2}\times\frac{1}{4}\times\frac{1}{6} = \frac{a^2b^2c^2}{6} \,//$$



24

$$\begin{cases} u = x \\ v = xy \\ w = 3z \end{cases} \Longrightarrow \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|^{-1} = \begin{vmatrix}1&0&0\\y&x&0\\0&0&3\end{vmatrix}^{-1} = \frac{1}{3x} = \frac{1}{3u}$$

$$\therefore \iiint_D (x^2y + 3xyz)\,dV = \int_1^2\int_0^2\int_0^3 \left(u^2\cdot\frac{v}{u} + 3u\cdot\frac{v}{u}\cdot\frac{w}{3}\right)\cdot\frac{1}{3u}\,dw\,dv\,du$$

$$= \int_1^2\int_0^2\int_0^3 \left(\frac{v}{3} + \frac{vw}{3u}\right) dw\,dv\,du$$

$$= \frac{1}{3}\left(\int_1^2 du\right)\left(\int_0^2 v\,dv\right)\left(\int_0^3 dw\right) + \frac{1}{3}\left(\int_1^2 \frac{du}{u}\right)\left(\int_0^2 v\,dv\right)\left(\int_0^3 w\,dw\right)$$

$$= 2 + 3\ln 2 \;/\!/$$