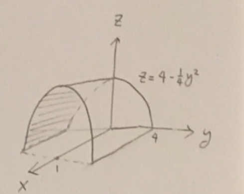
Solutions to HW 10

§16.6

Q14 Parametrize the surface S by

$(u,v) \mapsto (u, v, 4-\frac{v^2}{4})$, $\begin{matrix} u\in[0,1] \\ v\in[-4,4]\end{matrix}$



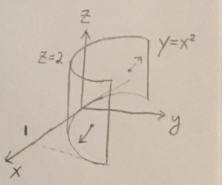
$$\Rightarrow \int_S G\, d\sigma = \int_0^1\int_{-4}^4 u\sqrt{v^2+4}\left\| \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & -\frac{v}{2}\end{vmatrix}\right\| dv\,du$$

$$= \int_0^1\int_{-4}^4 u\sqrt{v^2+4}\cdot\sqrt{(\tfrac{v}{2})^2+1}\, dv\,du$$

$$= \frac{1}{2}\times\left[\frac{1}{2}\left(\frac{v^3}{3}+4v\right)\right]_{-4}^{4} = \frac{56}{3}$$

Q20 Parametrize the surface S by

$(u,v)\mapsto(u,u^2,v)$ $u\in[-1,1], v\in[0,2]$



$$\begin{vmatrix} i & j & k \\ 1 & 2u & 0 \\ 0 & 0 & 1\end{vmatrix} = (2u,-1,0)$$

$\Rightarrow \vec{n} = \frac{1}{\sqrt{4u^2+1}}(2u,-1,0)$ (check $\vec{n}$ indeed points away from yz-plane)

$$\Rightarrow \int_S F\cdot\vec{n}\, d\sigma = \int_{-1}^1\int_0^2 (0,u^2,-uv)\cdot(2u,-1,0)\, dv\,du$$

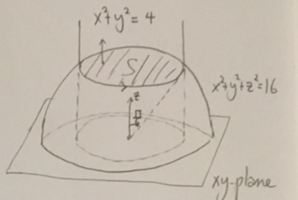
$$= \left[\frac{-u^3}{3}\right]_{-1}^{1}\cdot 2 = \frac{-4}{3}$$

§16.7

Q6 Parametrize the shaded S by

$(\theta,\phi)\mapsto(4\sin\phi\cos\theta, 4\sin\phi\sin\theta, 4\cos\phi)$

$\phi\in[0,\frac{\pi}{6}]$, $\theta\in[0,2\pi]$



$$\begin{vmatrix} i & j & k \\ -4\sin\phi\sin\theta & 4\sin\phi\cos\theta & 0 \\ 4\cos\phi\cos\theta & 4\cos\phi\sin\theta & -4\sin\phi\end{vmatrix} = (-16\sin^2\phi\cos\theta, -16\sin^2\phi\sin\theta, -16\sin\phi\cos\phi)$$

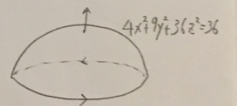
$\Rightarrow \vec{n} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ (outward of the hemi-sphere)

$$\Rightarrow \int_{\partial S} F\cdot dr = \iint_S \nabla\times F\cdot\vec{n}\, d\sigma \qquad \left(\nabla\times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2y^3 & 1 & z\end{vmatrix} = (0,0,-3x^2y^2)\right)$$

$$= -\int_0^{2\pi}\int_0^{\frac{\pi}{6}} 3(4\sin\phi\cos\theta)^2(4\sin\phi\sin\theta)^2(16\sin\phi\cos\phi)\, d\phi\, d\theta$$

$$= -512\pi\left[\sin^6\phi\right]_0^{\frac{\pi}{6}}$$

$$= -8\pi$$



Q7 $\iint_S \nabla\times F\cdot\vec{n}\, d\sigma = \int_{\partial S} F\cdot dr$

$$= \int_0^{2\pi}(2\sin t, 9\cos^2 t, 5\sin t)\cdot(-3\sin t, 2\cos t, 0)\, dt$$

$$= \int_0^{2\pi}(-6\sin^2 t + 18\cos^3 t)\, dt$$

$$= -6\pi$$

Q21 The unit normal of the plane pointing towards you is $\frac{1}{3}(2,2,1)$

$$\oint_C 2y\,dx + 3z\,dy - x\,dz = \iint_S \nabla\times F\cdot\vec{n}\, d\sigma \quad (S = \text{region enclosed by } C)$$

$$= \iint_S (-3,1,-2)\cdot\tfrac{1}{3}(2,2,1)\, d\sigma \qquad \left(\nabla\times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2y & 3z & -x\end{vmatrix} = (-3,1,-2)\right)$$

$$= -2\iint_S d\sigma = -2\,\text{Area}(S)$$

§16.8

Q7 $$\iint_{\partial D} F\cdot\vec{n}\, d\sigma = \iiint_D \nabla\cdot F\, dV = \int_0^{2\pi}\int_0^2\int_0^{r^2}(r\cos\theta - 1)\, r\, dz\, dr\, d\theta$$

$$= 2\pi\times\left[-\frac{r^4}{4}\right]_0^2 = -8\pi$$

Q15 $$\iint_{\partial D} F\cdot\vec{n}\, d\sigma = \iiint_D \nabla\cdot F\, dV = \int_0^{2\pi}\int_0^{\pi}\int_1^{\sqrt{2}}\left(15x^2+12y^2+3y^2+\cancel{e^y\sin z}+15z^2-\cancel{e^y\sin z}\right)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 15\left(\int_0^{2\pi}d\theta\right)\left(\int_0^{\pi}\sin\phi\, d\phi\right)\left(\int_1^{\sqrt{2}}\rho^4\, d\rho\right)$$

$$= 12\pi(4\sqrt{2}-1)$$

Q17 a) $$\nabla\cdot\nabla\times G = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z}-\frac{\partial P}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) = 0$$

b) $$\underset{\substack{S \\ \uparrow \\ \text{closed}}}{\oiint} \nabla\times G\cdot\vec{n}\, d\sigma = \iiint_D \nabla\cdot\nabla\times G\, dV \overset{(a)}{=} 0$$

(Here, we've used the fact that a closed surface $\subset \mathbb{R}^3$ is oriented and bounds a region)

Q29 $$\iint_S f\nabla g\cdot\vec{n}\, d\sigma = \iiint_D \nabla\cdot(f\nabla g)\, dV$$

Now $\nabla\cdot f\nabla g = \sum_{i=1}^3 \partial_i[f(\partial_i g)] = \sum_{i=1}^3[(\partial_i f)(\partial_i g) + f(\partial_i^2 g)]$

$$= \nabla f\cdot\nabla g + f\nabla\cdot\nabla g$$

$\therefore$ ans $= \iiint_D (f\nabla^2 g + \nabla f\cdot\nabla g)\, dV$ (1)

Q30 By Q29, we get another similar formula:

$$\iint_S g\nabla f\cdot\vec{n}\, d\sigma = \iiint_D (g\nabla^2 f + \nabla g\cdot\nabla f)\, dV \quad (2)$$

(1) − (2): $\iint_S (f\nabla g - g\nabla f)\cdot\vec{n}\, d\sigma$

$$= \iiint_D (f\nabla^2 g + \nabla f\cdot\nabla g - g\nabla^2 f - \nabla g\cdot\nabla f)\, dV$$

$$= \iiint_D (f\nabla^2 g - g\nabla^2 f)\, dV$$