Similarly, if ω is a 1-form and ξ is a 2-form,

then we can define $\omega \wedge \xi$.

eg: If $\omega = dx$, $\xi = dy \wedge dz$,

then
$$\omega \wedge \xi = dx \wedge (dy \wedge dz)$$
$$= dx \wedge dy \wedge dz$$

Note that we insist on the anti-commutativity of wedge product, we have

$$\begin{aligned} dx \wedge dy \wedge dz &= -dy \wedge dx \wedge dz \\ &= dy \wedge dz \wedge dx \\ &= -dz \wedge dy \wedge dx \\ &= dz \wedge dx \wedge dy \\ &= -dx \wedge dz \wedge dy \end{aligned}$$

And one can see that, as $\dim \mathbb{R}^3 = 3$, all "linear combinations" are just

$$f \, dx \wedge dy \wedge dz$$

which is called a 3-form (also called a volume form if $f > 0$)

Note: It is convenient to call smooth functions f the differential 0-form.

Summary (in $\mathbb{R}^3$)

0-form : f

1-form : $\omega_1 dx + \omega_2 dy + \omega_3 dz$

2-form : $\xi_1 dy \wedge dz + \xi_2 dz \wedge dx + \xi_3 dx \wedge dy$

3-form : $g \, dx \wedge dy \wedge dz$

(where f, g, ω_i, ξ_i are functions)

Note: On can certainly define k-form for any $k \geq 0$.

But in $\mathbb{R}^3$, k-forms are all zero for $k > 3$:

$dx^i \wedge dx \wedge dy \wedge dz = 0$, where $dx^i = dx, dy,$ or dz.

Change of Variables Formula : ($\mathbb{R}^2$)

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

$$\Rightarrow \begin{cases} dx = x_u\,du + x_v\,dv \\ dy = y_u\,du + y_v\,dv \end{cases}$$

$$\Rightarrow dx \wedge dy = (x_u\,du + x_v\,dv) \wedge (y_u\,du + y_v\,dv)$$

$$= (x_u y_v - x_v y_u)\,du \wedge dv$$

$$= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du \wedge dv$$

$$\boxed{dx \wedge dy = \frac{\partial(x,y)}{\partial(u,v)}\,du \wedge dv}$$

Hence, naturally

$$\boxed{\iint f(x,y)\,dx \wedge dy = \iint f(x(u,v),y(u,v))\,\frac{\partial(x,y)}{\partial(u,v)}\,du \wedge dv}$$

Compare with

$$\boxed{\iint f(x,y)\,dx\,dy = \iint f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

we can interpret "$dx \wedge dy$" as the "oriented" area element.

Similarly for
$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$$

(Ex!) $$\boxed{dx \wedge dy \wedge dz = \frac{\partial(x,y,z)}{\partial(u,v,w)}\, du \wedge dv \wedge dw}$$

(using $dx = x_u du + x_v dv + x_w dw, \dots$)

(Oriented Change of Variables Formula)

Exterior differentiation "d" on a form "ω"

0-form f	df (1-form)
1-form $\omega = \omega_1 dx + \omega_2 dy + \omega_3 dz$	$d\omega = d\omega_1 \wedge dx + d\omega_2 \wedge dy + d\omega_3 \wedge dz$ (2-form)
2-form $\xi = \xi_1 dy \wedge dz + \xi_2 dz \wedge dx + \xi_3 dx \wedge dy$	$d\xi = d\xi_1 \wedge dy \wedge dz + d\xi_2 \wedge dz \wedge dx + d\xi_3 \wedge dx \wedge dy$ (3-form)

3-form	$f\,dx\wedge dy\wedge dz$	$df\wedge dx\wedge dy\wedge dz=0$ (4-form)

eg 1 (in $\mathbb{R}^2$) $\omega = M\,dx + N\,dy$ $\left(\begin{array}{l} M=M(x,y) \\ N=N(x,y) \end{array}\right)$

then
$$\begin{aligned} d\omega &= dM\wedge dx + dN\wedge dy \\ &= (M_x dx + M_y dy)\wedge dx + (N_x dx + N_y dy)\wedge dy \\ &= (N_x - M_y)\,dx\wedge dy \end{aligned}$$

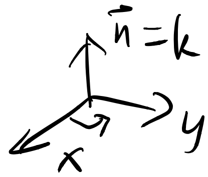
In this notation, Green's Thm can be written as

$$\oint_C \omega = \iint_R d\omega .$$

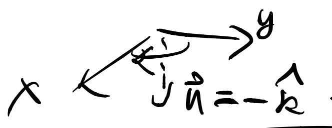
Note that $d\omega = (N_x - M_y)\,dx\wedge dy$ ← is oriented.

If we let $\vec{F} = M\hat{i} + N\hat{j} \leftrightarrow \omega = M\,dx + N\,dy$

then $(\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dA = (N_x - M_y)\underbrace{\hat{k}\cdot\vec{n}\,dA}_{dx\wedge dy} = d\omega$



Hence $\hat{k}\cdot\vec{n}\,dA = \begin{cases} dx\wedge dy, & \text{if } \vec{n}=\hat{k} \\ dy\wedge dx, & \text{if } \vec{n}=-\vec{k} \end{cases}$



eg 2 $\zeta = \zeta_1 \, dy \wedge dz + \zeta_2 \, dz \wedge dx + \zeta_3 \, dx \wedge dy$

then $d\zeta = d\zeta_1 \wedge dy \wedge dz + d\zeta_2 \wedge dz \wedge dx + d\zeta_3 \wedge dx \wedge dy$

$$= \left(\frac{\partial \zeta_1}{\partial x} dx + \cdots\right) \wedge dy \wedge dz$$

$$+ \left(\cdots + \frac{\partial \zeta_2}{\partial y} dy + \cdots\right) \wedge dz \wedge dx$$

$$+ \left(\cdots + \frac{\partial \zeta_3}{\partial z} dz\right) \wedge dx \wedge dy$$

$$= \left(\frac{\partial \zeta_1}{\partial x} + \frac{\partial \zeta_2}{\partial y} + \frac{\partial \zeta_3}{\partial z}\right) dx \wedge dy \wedge dz$$

$$= (\operatorname{div} \vec{F}) \, dx \wedge dy \wedge dz$$

where $\vec{F} = \zeta_1 \hat{i} + \zeta_2 \hat{j} + \zeta_3 \hat{k}$

Hence the divergence theorem, we can be written

as $\displaystyle \iiint_D d\zeta = \iiint_D \left(\frac{\partial \zeta_1}{\partial x} + \frac{\partial \zeta_2}{\partial y} + \frac{\partial \zeta_3}{\partial z}\right) dx \wedge dy \wedge dz$

$$= \iiint_D \operatorname{div} \vec{F} \, dV$$

$$= \iint_{S = \partial D} \vec{F} \cdot \vec{n} \, d\sigma$$

To see the relation between $\vec{F}\cdot\vec{n}\,d\sigma$ and ζ, we parametrize S:

$$\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$$

$$\Rightarrow \begin{cases} \vec{r}_u = x_u\hat{i} + y_u\hat{j} + z_u\hat{k} \\ \vec{r}_v = x_v\hat{i} + y_v\hat{j} + z_v\hat{k} \end{cases}$$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix}\hat{i} + \begin{vmatrix} z_u & z_v \\ x_u & x_v \end{vmatrix}\hat{j} + \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}\hat{k}$$

If $\vec{r}_u \times \vec{r}_v$ is <u>outward</u>, then

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \quad \text{and} \quad d\sigma = |\vec{r}_u \times \vec{r}_v|\,du\,dv$$

$$= |\vec{r}_u \times \vec{r}_v|\,du\wedge dv$$

(correct orientation)

$$\therefore \vec{F}\cdot\vec{n}\,d\sigma = \vec{F}\cdot\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}|\vec{r}_u \times \vec{r}_v|\,du\wedge dv$$

$$= \left(\zeta_1\frac{\partial(y,z)}{\partial(u,v)} + \zeta_2\frac{\partial(z,x)}{\partial(u,v)} + \zeta_3\frac{\partial(x,y)}{\partial(u,v)}\right)du\wedge dv$$

$$= \zeta_1\,dy\wedge dz + \zeta_2\,dz\wedge dx + \zeta_3\,dx\wedge dy = \zeta$$

$$\therefore \boxed{\iiint_D d\xi = \iint_{S=\partial D} \xi}$$

<u>eg3</u> <u>Stokes' Thm</u>

If $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k} \Leftrightarrow \omega = M\,dx + N\,dy + L\,dz$

Then $d\omega = (L_y - N_z)\,dy\wedge dz + (M_z - L_x)\,dz\wedge dx$ (Ex!)

$+ (N_x - M_y)\,dx\wedge dy$

Compare to

$$\vec{\nabla}\times\vec{F} = (L_y - N_z)\hat{i} + (M_z - L_x)\hat{j} + (N_x - M_y)\hat{k}$$

$$\therefore \quad d\omega = (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,d\sigma$$

Stokes' Thm $\Leftrightarrow$

$$\oint_{C=\partial S} \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,d\sigma$$

$$\downarrow \qquad\qquad \downarrow$$

$$\boxed{\oint_{C=\partial S} \omega = \iint_S d\omega}$$

Generalization to manifold of n-dimension with boundary

- M = n dim'l Manifold (oriented)
- ∂M = boundary (oriented with induced orientation)
- ω = $(n-1)$-form on M (smooth)

Then

$$\boxed{\underbrace{\int_M d\omega}_{n\text{-dim'l integral}} = \underbrace{\int_{\partial M} \omega}_{(n-1)\text{-dim'l integral}}}$$

Note: ∂M is always closed, i.e. no boundary.

$$\therefore \boxed{\partial(\partial M) = \partial^2 M = 0}$$

boundary has no boundary

∂S is a closed curve

Hence if $\omega = d\eta$, for some $(n-2)$-form η, then

then $\int_M d(d\eta) = \int_M d\omega = \int_{\partial M} \omega$

$= \int_{\partial M} d\eta = \int_{\partial(\partial M)} \eta = 0$ (for any η.)

This suggests $\boxed{d^2\eta = 0}$, $\forall$ differential form

Ex: Verify this for 0-form and 1-form in $\mathbb{R}^3$
and observes that these are just

$$\begin{cases} \vec{\nabla} \times \vec{\nabla} f = 0 & (d^2 f = 0) \\ \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0 & (d^2\omega = 0) \end{cases}$$

eg: Let $\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$

Check: $d\omega = 0$

But $\omega \neq df$ for any smooth function on $\mathbb{R}^2 \setminus \{(0,0)\}$

(Since $\omega = d\theta$ and θ is not defined on $\mathbb{R}^2 \setminus \{(0,0)\}$)

Hence $\quad d\omega = 0 \not\Rightarrow \omega = d\eta$ in general

$(\Leftarrow)$

yes

<u>Note</u>: <u>Thm 10</u> can be written as:

$\Omega \subset \mathbb{R}^2$ simply-connected, then

$d\omega = 0 \Leftrightarrow \omega = df$ for some smooth function f on Ω.