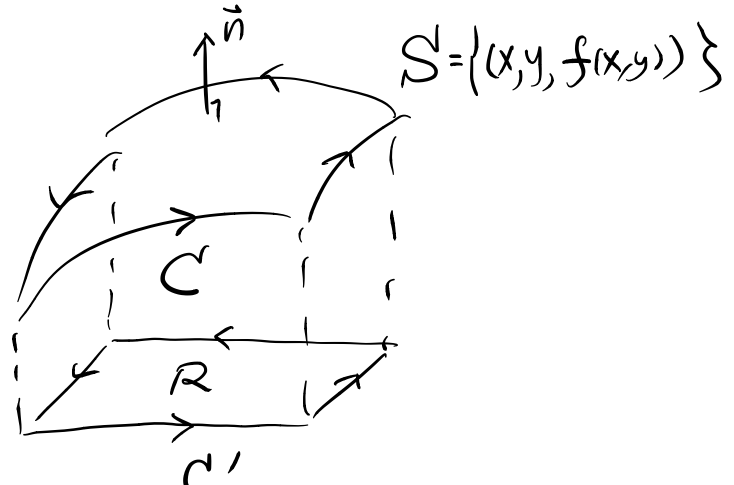
Proof of Stokes' Thm

Special case : $\tilde{S}$ is a graph given by

$z = f(x,y)$ over a region R with upward normal



Assume C is boundary $\tilde{S}$, and C' is the boundary of R (anti-clockwise wrt $\tilde{S}$ & R respectively)

Parametrize the graph as

$$\vec{r}(x,y) = x\hat{i} + y\hat{j} + f(x,y)\hat{k} \,, \quad (x,y) \in R$$

Then
$$\begin{cases} \vec{r}_x = \hat{i} + \dfrac{\partial f}{\partial x}\hat{k} \\ \vec{r}_y = \hat{j} + \dfrac{\partial f}{\partial y}\hat{k} \end{cases}$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}$$

upwark

hence $\vec{n} = \dfrac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|}$ is the upward normal.

and $d\sigma = |\vec{r}_x \times \vec{r}_y| \underbrace{dx\,dy}_{} = |\vec{r}_x \times \vec{r}_y| dA$

$\hookrightarrow$ dA of the region R

Let $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ be the C^1 vector field.

Then
$$\iint_S \vec{\nabla} \times \vec{F} \cdot \vec{n}\, d\sigma = \iint_R \vec{\nabla} \times \vec{F}(\vec{r}(x,y)) \cdot \frac{\vec{r}_x \times \vec{r}_y}{\cancel{|\vec{r}_x \times \vec{r}_y|}} \cdot \cancel{|\vec{r}_x \times \vec{r}_y|}\, dA$$

$$= \iint_R \left[(L_y - N_z)\hat{i} + (M_z - L_x)\hat{j} + (N_x - M_y)\hat{k} \right] \cdot \left[-f_x \hat{i} - f_y \hat{j} + \hat{k} \right] dA$$

$$= \iint_R \left[-f_x (L_y - N_z) - f_y (M_z - L_x) + (N_x - M_y) \right] dx\,dy.$$

For the line integral

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C M\,dx + N\,dy + L\,dz$$

$$= \oint_{C'} M\,dx + N\,dy + L\,df \qquad (z = f(x,y))$$

when restrict on the project $= \oint_{C'} M\,dx + N\,dy + L(f_x\,dx + f_y\,dy)$

$$= \oint_{C'} (M + Lf_x)\,dx + (N + Lf_y)\,dy$$

Reason: If C' is parametrized by

$$\vec{r}(t) = (x(t), y(t)) \quad \text{for } a \le t \le b,$$

then C is parametrized by

$$\vec{F}(t) = (x(t), y(t), f(x(t), y(t)))$$

$$= x(t)\,\hat{i} + y(t)\,\hat{j} + f(x(t), y(t))\,\hat{k}.$$

$$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = \int_a^b M(\vec{F}(t))\,x'(t)\,dt + N(\vec{F}(t))\,y'(t)\,dt + L(\vec{F}(t))\,\frac{d}{dt}\big(f(x(t), y(t))\big)\,dt$$

$$= \int_a^b [Mx' + Ny' + L(f_x x' + f_y y')]\,dt$$

$$= \int_a^b [(M + Lf_x)x' + (N + Lf_y)y']\,dt$$

$$= \int_{C'} (M + Lf_x)\,dx + (N + Lf_y)\,dy$$

Then by Green's Theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_{C'} (M + L f_x)\,dx + (N + L f_y)\,dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(N + L f_y) - \frac{\partial}{\partial y}(M + L f_x)\right] dA$$

$$= \iint_R \left\{ \begin{array}{l} \frac{\partial}{\partial x}\left[N(x,y,f(x,y)) + L(x,y,f(x,y))\, f_y(x,y)\right] \\ \quad - \frac{\partial}{\partial y}\left[M(x,y,f(x,y)) + L(x,y,f(x,y))\, f_x(x,y)\right] \end{array} \right\} dx\,dy$$

$$= \iint_R \left[\begin{array}{l} (N_x + N_z f_x) + (L_x + \cancel{L_z f_x}) f_y + \cancel{L f_{yx}} \\ -(M_y + M_z f_y) - (L_y + \cancel{L_z f_y}) f_x - \cancel{L f_{xy}} \end{array} \right] dx\,dy$$

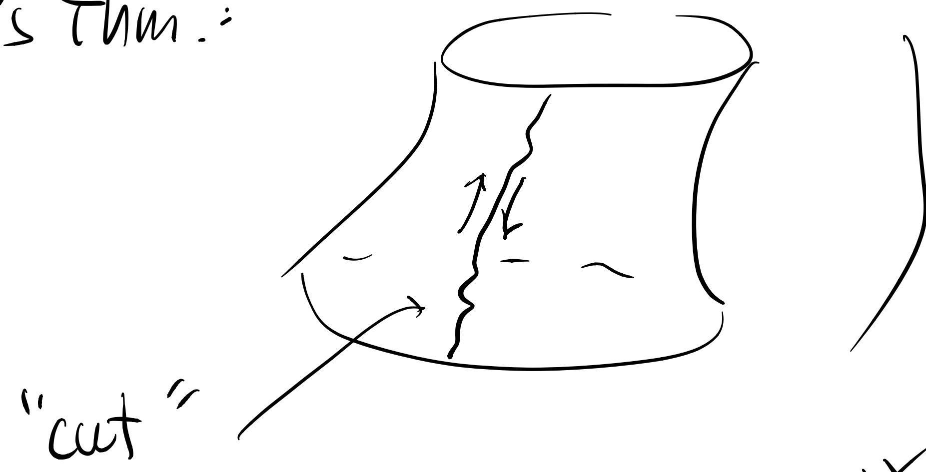
(provided your surface is C^2)

$$= \iint_R \left[-f_x (L_y - N_z) - f_y (M_z - L_x) + (N_x - M_y) \right] dx\,dy$$

i.e. $$\iint_S \vec{\nabla} \times \vec{F} \cdot \vec{n}\, d\sigma = \oint_C \vec{F}\cdot d\vec{r}.$$

This completes the case of (C^2) graph.

general case : Divides S into finite many pieces which are graph in certain projection. (This includes S with many boundary components as in Green's Thm.:



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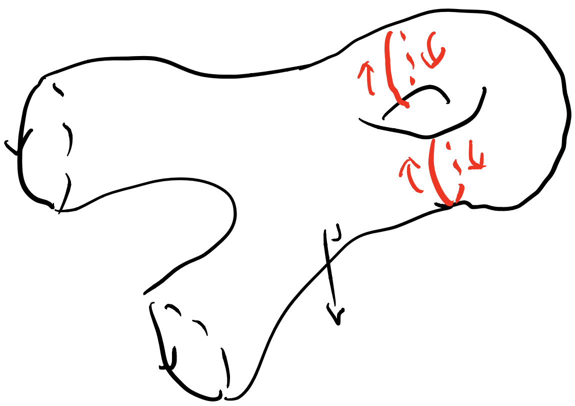
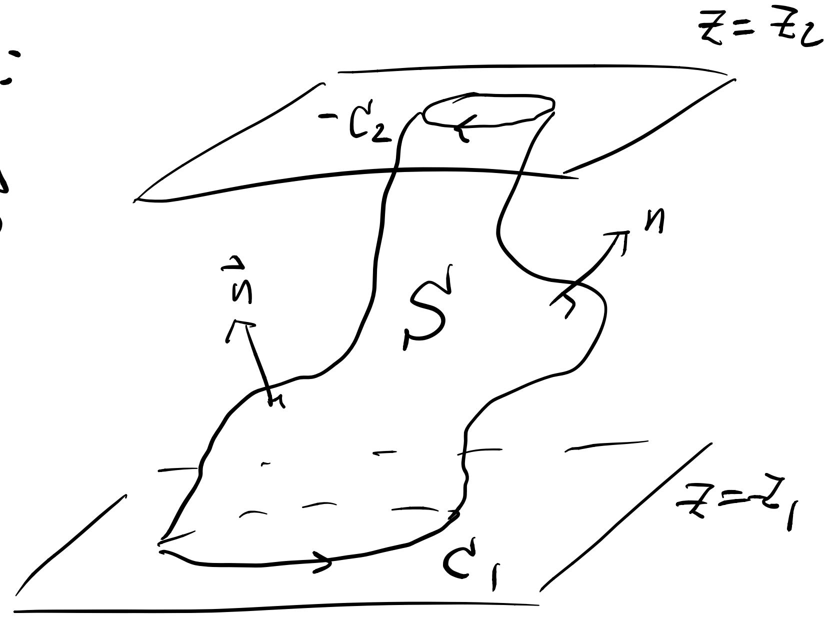
Note: Stokes' Thm applies to surfaces like the following one :

cuts along the close curves in "red".



eg62: Let $\vec{F}$ be a vector field such that $\vec{\nabla}\times\vec{F}=0$ and defined on a region containing the surface S with unit normal vector field $\vec{n}$ as in the figure:



The boundary C of S has 2 components C_1 and C_2 at the level $z=z_1$ and $z=z_2$ respectively.

If both C_1 and C_2 oriented anticlockwise with respect to the "horizontal planes"

Then when C oriented with respect to $\vec{n}$,

then $$C = C_1 - C_2 .$$

And Stokes' Thm $\Rightarrow$

$$0 = \iint_S \vec{\nabla}\times\vec{F}\cdot\vec{n}\,d\sigma = \oint_{C} \vec{F}\cdot d\vec{r}$$

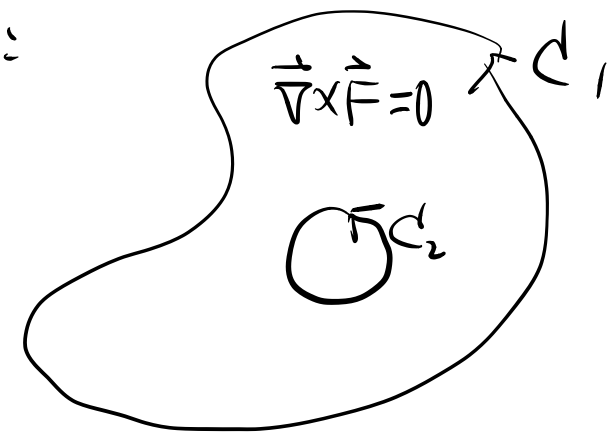
$C \leftarrow$ oriented wrt $\vec{n}$

$$= \oint_{C_1} \vec{F}\cdot d\vec{r} - \oint_{C_2} \vec{F}\cdot d\vec{r}$$

oriented wrt the "horizontal plane"

$$\Rightarrow \quad \oint_{C_1} \vec{F}\cdot d\vec{r} = \oint_{C_2} \vec{F}\cdot d\vec{r}.$$

Compare this with Green's Thm on plane region with one hole :



$$\Rightarrow \quad \oint_{C_1} \vec{F}\cdot d\vec{r} = \oint_{C_2} \vec{F}\cdot d\vec{r} \qquad \text{(Ex!)}$$

anti-clockwise wrt "plane".

Proof of Thm 10

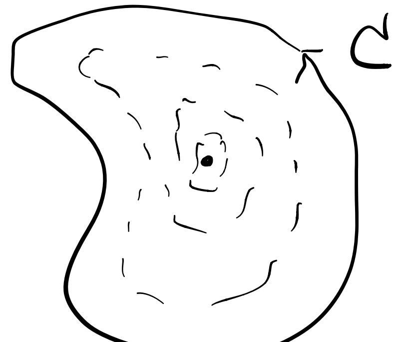
Only the "$\Leftarrow$" part remains to be proved:

By assumption $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ satisfies the system of eqts. in the Cor. to the Thm 9, that is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{\partial L}{\partial y} \quad \text{and} \quad \frac{\partial L}{\partial x} = \frac{\partial M}{\partial z}.$$

Hence $\vec{\nabla} \times \vec{F} = 0$

Let C be a simple closed curve in a simply-connected region D. Then C be deformed to a point inside D. The process of deformation gives an oriented surface $S' \subset D$ such that the boundary of $S' = C$.



By Stokes' Thm,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_{S'} \vec{\nabla} \times \vec{F} \cdot \vec{n} \, d\sigma = 0 \quad (\vec{\nabla} \times \vec{F} = 0)$$

Then Thm 9 $\Rightarrow$ $\vec{F}$ is conservative. #

Summary

$n=2$	$n=3$
Tangential form of Green's Thm $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \vec{\nabla}\times\vec{F}\cdot\hat{k}\, dA$	Stokes' Thm $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot\vec{n}\, d\sigma$
Normal form of Green's Thm $\oint_C \vec{F}\cdot\vec{n}\, ds = \iint_R \vec{\nabla}\cdot\vec{F}\, dA$	Divergence Thm $\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iiint_D \vec{\nabla}\cdot\vec{F}\, dV$ closed surface = boundary of D.

(see below)

Thm 13 (Divergence Theorem)

(no boundary)

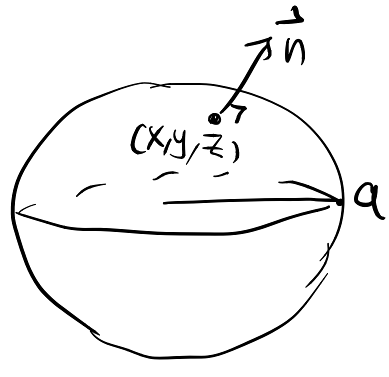
Let $\vec{F}$ be a C^1 vector field on $\Omega \overset{\text{open}}{\subseteq} \mathbb{R}^3$,

S be a piecewise smooth orientable closed surface enclosing a (solid) region $D \subseteq \Omega$.

Let $\vec{n}$ be the outward pointing unit normal vector field on S. Then

$$\boxed{\begin{aligned} \iint_S \vec{F}\cdot\vec{n}\, d\sigma &= \iiint_D \operatorname{div}\vec{F}\, dV \\ &= \iiint_D \vec{\nabla}\cdot\vec{F}\, dV \end{aligned}}$$

eg 63 Verify Divergence Thm for



$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$,

$S: x^2+y^2+z^2 = a^2$, $a>0$

sphere

($\Rightarrow D =$ solid sphere bounded by S)

Soln : At $(x,y,z) \in S$

$$\vec{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{1}{a}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iint_S (x\hat{i}+y\hat{j}+z\hat{k})\cdot\frac{1}{a}(x\hat{i}+y\hat{j}+z\hat{k})\,d\sigma$$

$$= \iint_S a\,d\sigma = a\iint_S d\sigma = a\,\underset{\uparrow\ 4\pi\,\text{radius}^2}{\text{Area}(S)}$$

$$= 4\pi a^3 \quad (\text{check!})$$

On the other hand

$$\text{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\therefore \iiint_D \text{div}\vec{F}\,dV = \iiint_D 3\,dV = 3\,\underset{\uparrow\ \frac{4}{3}\pi(\text{radius})^3}{\text{Vol}(D)}$$

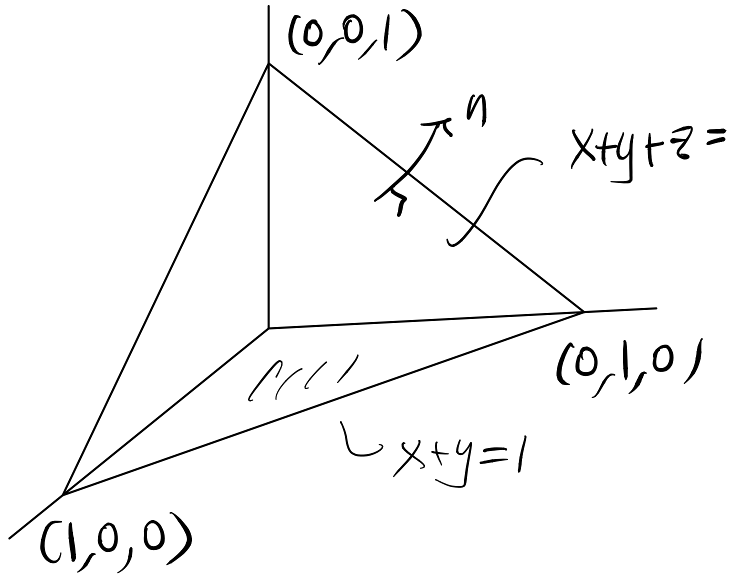
$$= 3\cdot\frac{4}{3}\pi a^3 = 4\pi a^3$$

$$= \iint_S \vec{F}\cdot\vec{n}\,d\sigma.$$

eg63: $\vec{F} = x\sin y\,\hat{i} + (\cos y + z)\hat{j} + z^2\hat{k}$

Compute outward flux of $\vec{F}$ across boundary ∂T of $T = \left\{(x,y,z)\in\mathbb{R}^3 : \begin{matrix} x+y+z\le 1 \\ x,y,z\ge 0\end{matrix}\right\}$

$$\iint_{\partial T} \vec{F}\cdot\vec{n}\,d\sigma$$



Soln:

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F}$$

$$= \frac{\partial}{\partial x}(x\sin y) + \frac{\partial}{\partial y}(\cos y + z) + \frac{\partial}{\partial z}(z^2)$$

$$= 2z \quad \text{(check!)}$$

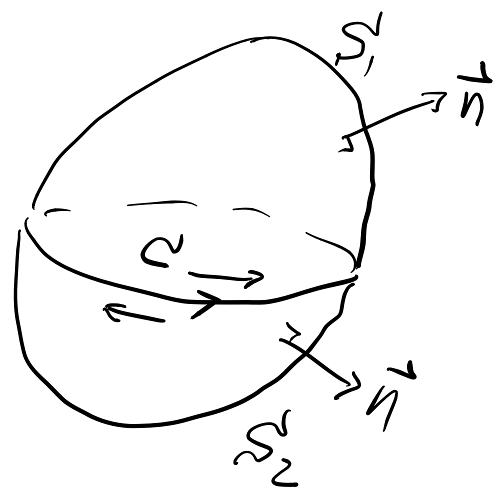
Divergence Thm

$$\Rightarrow \iint_{\partial T}\vec{F}\cdot\vec{n}\,d\sigma = \iiint_T \operatorname{div}\vec{F}\,dV$$

$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y} 2z\,dz\,dy\,dx$$

$$= \frac{1}{12} \quad \text{(check!)}$$

eg64: Let S_1, S_2 be 2 surfaces with common boundary curve C such that $S_1 \cup S_2$ forms a close surface enclosing a solid region D (without hole)



Suppose $\vec{n}$ is the outward normal of the solid region D.

Then the orientation of C wrt $(S_1, \vec{n})$ and $(S_2, \vec{n})$ are opposite.

Stokes' Thm $\Rightarrow$

$$\iint_{S_1} \vec{\nabla}\times\vec{F}\cdot\vec{n}\, d\sigma = \oint_C \vec{F}\cdot d\vec{r}$$

(+ve oriented wrt $(S_1, \vec{n})$)

$$= -\oint_C \vec{F}\cdot d\vec{r}$$

(+ve oriented wrt $(S_2, \vec{n})$)

$$= -\iint_{S_2} \vec{\nabla}\times\vec{F}\cdot\vec{n}\, d\sigma$$

$$\Rightarrow \iint_{S_1' \cup S_2'} \vec{\nabla} \times \vec{F} \cdot \vec{n} \, d\sigma = 0$$

Divergence Thm $\Rightarrow$

$$\iiint_D \operatorname{div}(\vec{\nabla} \times \vec{F}) \, dV = \iint_{S_1' \cup S_2'} (\vec{\nabla} \times \vec{F}) \cdot \vec{n} \, d\sigma = 0$$

(true for any C^2 vector fields $\vec{F}$ defined on D.)

It is consistent with $\boxed{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0}$ (Ex!)

$\forall$ C^2 vector field!

i.e. $\boxed{\operatorname{div}(\operatorname{curl} \vec{F}) = 0}$ $\quad (\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0)$

Compare $\boxed{\operatorname{curl}(\operatorname{grad} f) = 0}$ $\quad (\vec{\nabla} \times (\vec{\nabla} f) = 0)$