Thm 12 (Stokes' Theorem)

Let S be a piecewise smooth oriented surface with piecewise smooth boundary C (including the case that C is a union of finite many curves). Let

$$\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$$ be a C^1 vector fields.

Suppose C is oriented anti-clockwisely with respect to the unit normal vector field $\vec{n}$ on S. Then

$$\boxed{\begin{aligned}\oint_C \vec{F}\cdot d\vec{r} &= \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,d\sigma \\ &= \iint_S \vec{\nabla}\times\vec{F}\cdot\vec{n}\,d\sigma\end{aligned}}$$

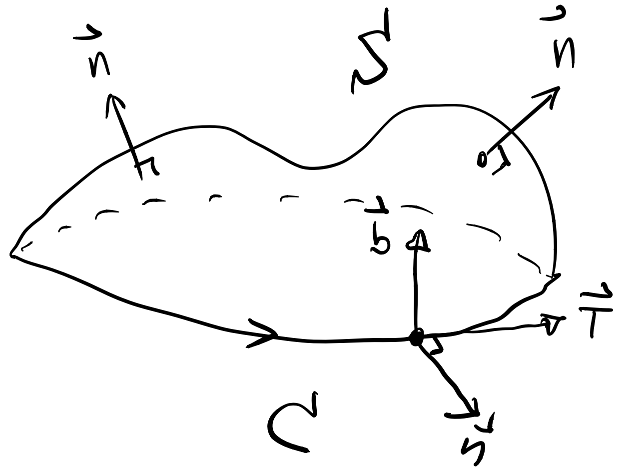
Here (i) if $C = C_1 \cup \cdots \cup C_k$, then it means

$$\sum_{i=1}^{k} \oint_{C_i} \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot\vec{n}\,d\sigma.$$

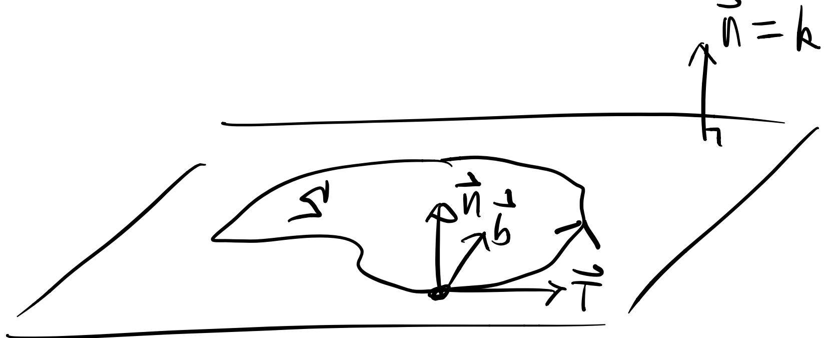
(ii) "C oriented wrt the unit normal vector field $\vec{n}$" means we choose the direction of C such that its tangent vector $\vec{T}$ satisfies:

$$\boxed{\vec{b} = \vec{n} \times \vec{T} \text{ pointing } \underline{\text{toward}} \text{ the surface } S.}$$

<u>eg</u> 60 (1)

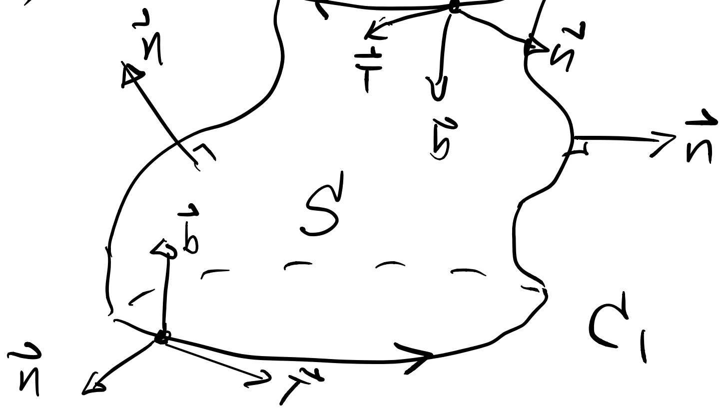


(2) If $S \subset \mathbb{R}^2$ with $\vec{n} = \hat{k}$



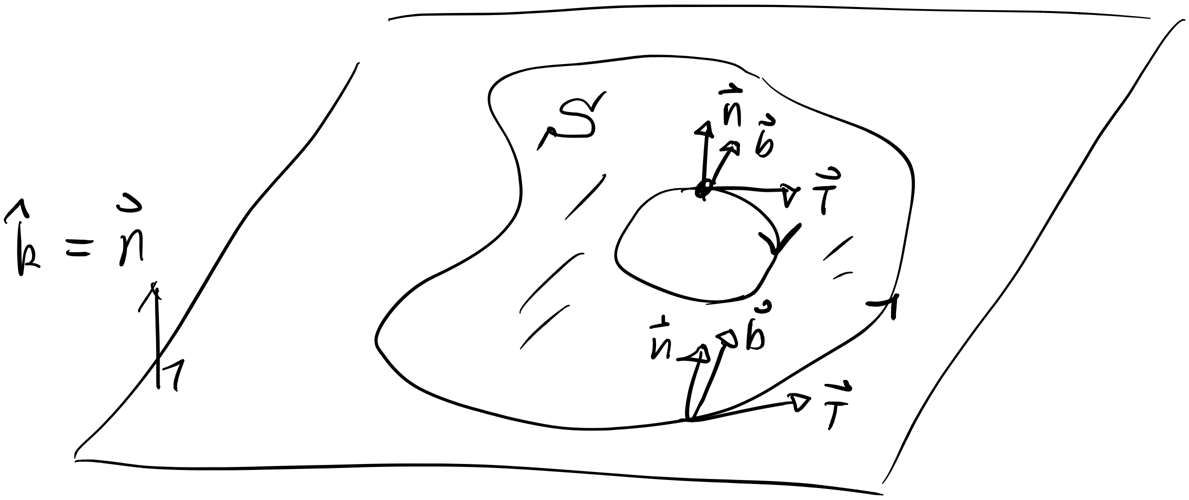
same as the anti-clockwise orientation in the plane

(3)

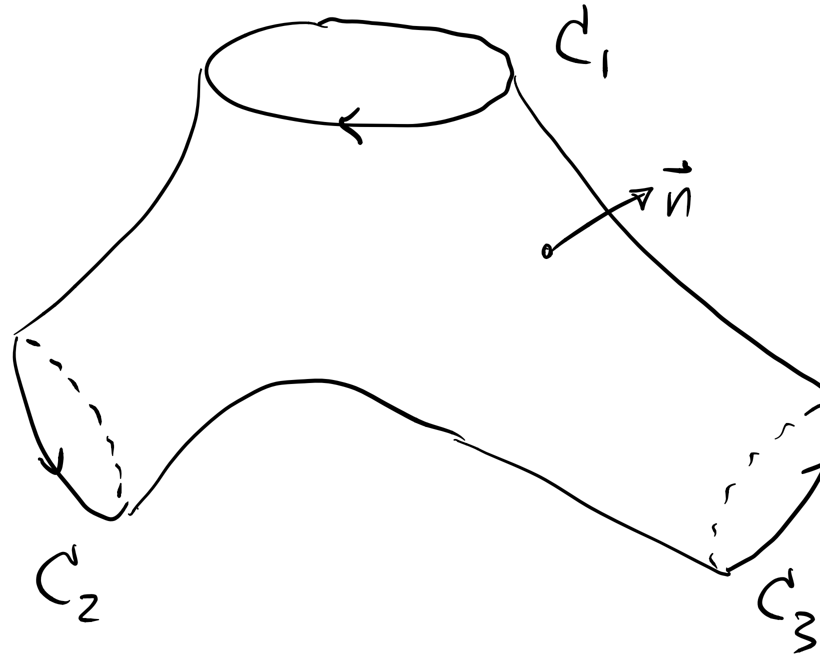


$C = C_1 \cup C_2$

(4)



(5)

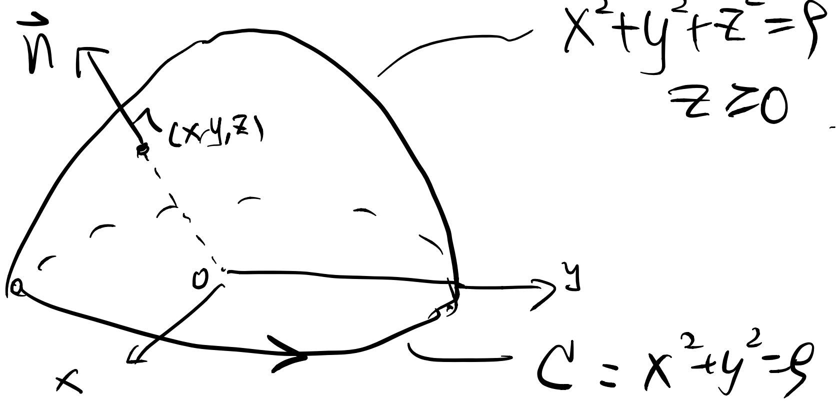


(Ex!)

eg6)

(a) S_1 : $x^2+y^2+z^2 = 9$, $z \geq 0$, with upward normal

boundary C : $x^2+y^2=9$, $z=0$



Let $\vec{F} = y\hat{i} - x\hat{j}$

Verifying Stokes' Thm :

$$C : \vec{r}(t) = (3\cos t, 3\sin t, 0) \qquad 0 \leq t \leq 2\pi$$

$$= 3\cos t\,\hat{i} + 3\sin t\,\hat{j}$$

$$d\vec{r} = (-3\sin t\,\hat{i} + 3\cos t\,\hat{j})\,dt$$

Along C, $\vec{F}(\vec{r}(t)) = y\hat{i} - x\hat{j}$

$$= 3\sin t\,\hat{i} - 3\cos t\,\hat{j}$$

$$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (3\sin t\,\hat{i} - 3\cos t\,\hat{j}) \cdot (-3\sin t\,\hat{i} + 3\cos t\,\hat{j})\,dt$$

$$= \int_0^{2\pi} -9\,dt = -18\pi \quad \text{(check!)}$$

For the surface integral :

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = -2\hat{k} \quad \text{(check!)}$$

Since S_1 is a hemisphere centered at origin,

$$\vec{n} = \frac{1}{3}(x\hat{i} + y\hat{j} + z\hat{k}) \quad \text{on } S_1$$

The surface S_1 can be regarded as level surface given by $g(x,y,z) = x^2+y^2+z^2 = 9$.

Note $\vec{\nabla} g = (2x, 2y, 2z)$

Since $z>0$, (except the boundary) on S_1,

$$\frac{\partial g}{\partial z} = 2z \neq 0$$

Hence $$d\sigma = \frac{|\vec{\nabla} g|}{\left|\frac{\partial g}{\partial z}\right|} dxdy = \frac{\sqrt{(2x)^2+(2y)^2+(2z)^2}}{|2z|} dxdy$$

$$= \frac{2\sqrt{x^2+y^2+z^2}}{2|z|} dxdy = \frac{3}{|z|} dxdy$$

$$= \frac{3}{z} dxdy$$

Therefore

$$\iint_{S_1} \vec{\nabla} \times \vec{F} \cdot \vec{n} \, d\sigma$$

$$= \iint_{x^2+y^2 \leq 9} (-2\hat{k}) \times \frac{1}{3}(x\hat{i}+y\hat{j}+z\hat{k}) \cdot \frac{3}{z} dxdy$$

(Check!) $\stackrel{}{=} \iint_{x^2+y^2\leq 9} (-2) dxdy = -18\pi$ (check!)

(b) S_2 :

$C : x^2+y^2=9, \quad z=0$

$$S_2 : x^2+y^2 \leq 9, \quad z=0$$

Same $\vec{F} = y\hat{i} - x\hat{j}$

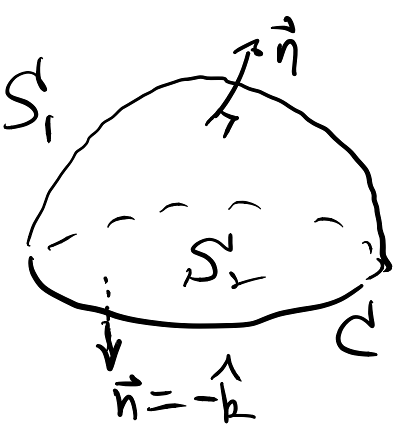
$$\iint_{S_2} \vec{\nabla}\times\vec{F} \cdot \vec{n}\, d\sigma = \iint_{x^2+y^2\leq 9} (-2\hat{k})\cdot\hat{k}\, dx\,dy$$

$$= -18\pi \quad \text{(same answer.)}$$

(c) $\vec{F} = y\hat{i} - x\hat{j}$ $\quad S_3 = S_1 \cup S_2$

S_1: outward normal $\quad$ S_2: with $-\hat{k}$

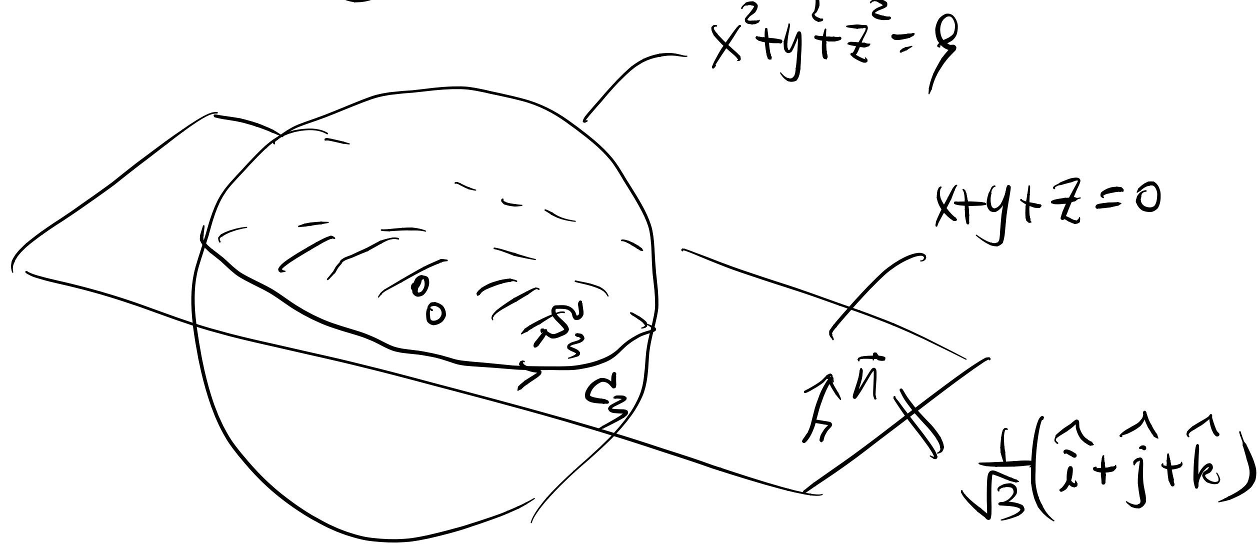
$\vec{n}$ = outward normal (of the solid bounded by S_3)



$$\iint_{S_3} (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, d\sigma = \iint_{S_1} \vec{\nabla}\times\vec{F}\cdot\vec{n}\, d\sigma - \iint_{S_2} \vec{\nabla}\times\vec{F}\cdot\hat{k}\, d\sigma$$

$$= 0.$$

(d) Same $\vec{F} = y\hat{i} - x\hat{j}$



$$S_4 = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2 \leq 9,\ x+y+z=0\}$$

Applying Stokes' Theorem

$$\oint_{C_3} \vec{F}\cdot d\vec{r} = \iint_{S_3} \vec{\nabla}\times\vec{F}\cdot\vec{n}\, d\sigma$$

$$= \iint_{S_3} (-2\hat{k})\cdot\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})\, d\sigma$$

check $$= -\frac{2}{\sqrt{3}}\iint_{S_3} d\sigma$$

$$= -\frac{2}{\sqrt{3}}\,\text{Area}(S_3)$$

$$= -\frac{2}{\sqrt{3}}(\pi 3^2) = -\frac{18\pi}{\sqrt{3}}$$

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