Def 16 Surface Integral (of a function)

Suppose $G: S \to \mathbb{R}$ is a continuous function on a surface S, parametrized by $\vec{r}(u,v)$, $(u,v) \in R$.

Then the integral of G on S is

$$\iint_S G \, d\sigma = \iint_R G(\vec{r}(u,v)) \, |\vec{r}_u \times \vec{r}_v| \, dA .$$

Note: In the cases of graph or level surface, we have

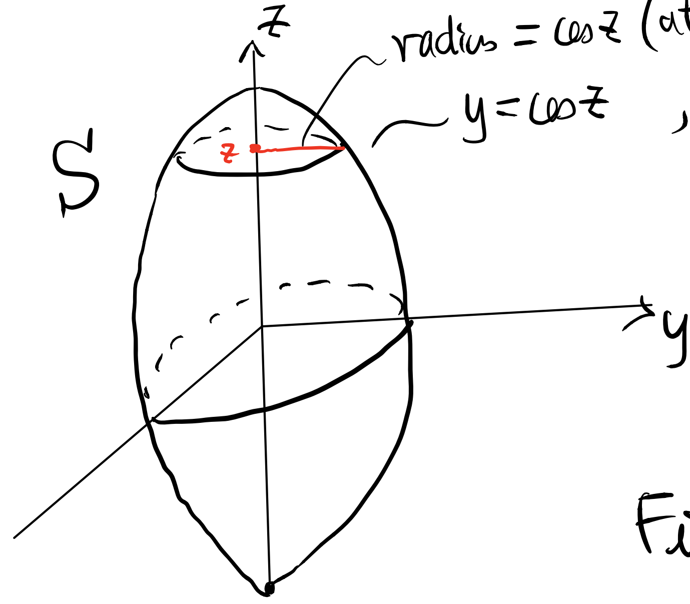
(i) $$\iint_S G \, d\sigma = \iint_{(x,y)} G(x,y,h(x,y)) \sqrt{1+(h_x)^2+(h_y)^2} \, dx\,dy$$

(for $z = h(x,y)$)

(ii) $$\iint_S G \, d\sigma = \iint_{(x,y)} G(x,y,z) \frac{|\nabla F|}{|F_z|} \, dx\,dy,$$

(for $F(x,y,z) = c$, $F_z \neq 0$)

eg 56 (a surface of revolution of the curve $y=\cos z$)



Let $G(x,y,z) = \sqrt{1-x^2-y^2}$ be function on S.

Find $\iint_S G\, d\sigma$.

Soln: S can be parametrized by

$$\begin{cases} x = \cos z \cos\theta & -\pi < \theta \le \pi \\ y = \cos z \sin\theta & , \quad -\frac{\pi}{2} \le z \le \frac{\pi}{2} \\ z = z \end{cases}$$

i.e. $\vec{r}(\theta, z) = \cos z \cos\theta\, \hat{i} + \cos z \sin\theta\, \hat{j} + z\, \hat{k}$

(Note: (θ, z) is not a parametrization for the whole surface as it is not 1-1 for $\theta = -\pi$ & $\theta = \pi$

and not smooth at $z = \pm\frac{\pi}{2}$.

However, the exceptional set is of "1-dim" and has no contribution to the surface integral.)

$$\begin{cases} \vec{r}_\theta = -\cos z \sin\theta\,\hat{i} + \cos z\cos\theta\,\hat{j} \\ \vec{r}_z = -\sin z\cos\theta\,\hat{i} - \sin z\cos\theta\,\hat{j} + \hat{k} \end{cases}$$

$$\Rightarrow \quad \vec{r}_\theta \times \vec{r}_z = \cos z\cos\theta\,\hat{i} + \cos z\sin\theta\,\hat{j} + \sin z\cos z\,\hat{k} \quad \text{(check!)}$$

$$\Rightarrow \quad |\vec{r}_\theta \times \vec{r}_z| = \sqrt{\cos^2 z\,(1+\sin^2 z)} = \cos z\sqrt{1+\sin^2 z}$$

$$\left(\text{since } \cos z \geq 0 \text{ for } -\tfrac{\pi}{2} \leq z \leq \tfrac{\pi}{2}\right)$$

Then $$\iint_S G\,d\sigma = \int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} G(\vec{r}(\theta,z))\,|\vec{r}_\theta \times \vec{r}_z|\,dz\,d\theta$$

$$= \int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-x^2-y^2}\;|\vec{r}_\theta \times \vec{r}_z|\,dz\,d\theta$$

$$= \int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\cos^2 z}\cdot\cos z\sqrt{1+\sin^2 z}\,dz\,d\theta$$

$$= \int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin z|\cos z\sqrt{1+\sin^2 z}\,dz\,d\theta$$

(check)

$$= 2\int_{-\pi}^{\pi}\int_{0}^{\frac{\pi}{2}} \sin z\cos z\sqrt{1+\sin^2 z}\,dz\,d\theta$$

$$= \frac{4\pi}{3}(2\sqrt{2}-1) \qquad \text{(Ex!)}$$

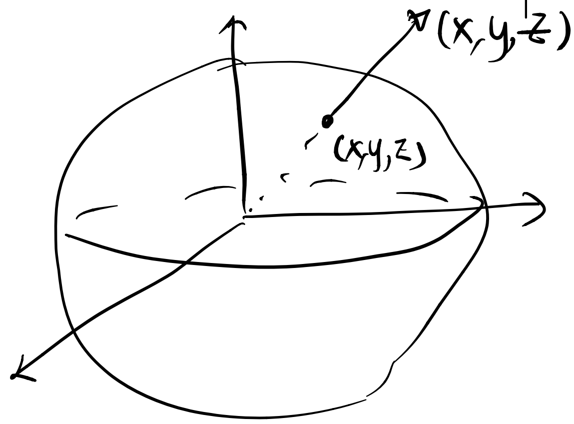
\#

To integral vector fields over a surface,

Def 17 (Orientation of a surface in $\mathbb{R}^3$)

A surface S is orientable if one can define a unit normal vector field continuously at every point of S.

eg 57 (i) $S^2 = \{x^2+y^2+z^2=1\}$

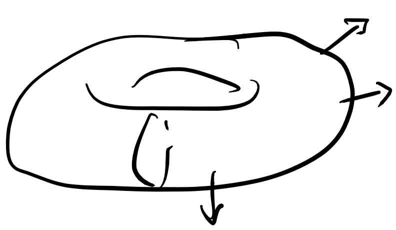


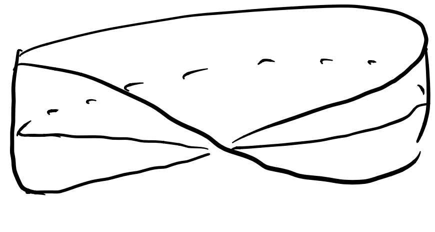
$$\vec{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$= x\hat{i}+y\hat{j}+z\hat{k} \text{ on } S^2$$

defines a continuous unit normal vector field on S^2

$\Rightarrow S^2$ is orientable.

(ii)  Torus is orientable

(iii)  Möbius strip is not orientable.

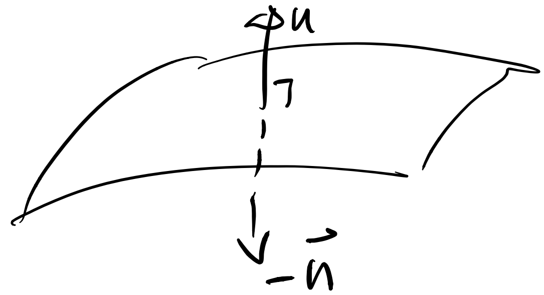
Remark: Parametric surface $S: \vec{r}(u,v)$ are always orientable:

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$ is a continuous unit normal vector field on S.

(provided $\vec{r}(u,v)$ is 1-1.)

Terminology:

Given a connected, orientable surface $S \subset \mathbb{R}^3$, there are two ways to assign the continuous unit normal vector field



Suppose S is orientable and we have chosen one continuous unit normal vector field $\vec{n}$. Then

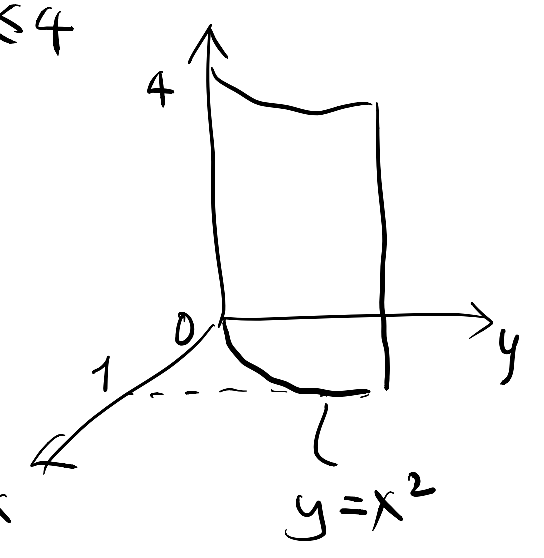
Def 18: We said that a parametrization $\vec{r}(u,v)$ of S is compatible with the orientation of S given by the unit normal vector field $\vec{n}$, if

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}.$$

Def 19: Let S be orientable with unit normal $\vec{n}$.
Let $\vec{F}$ be a vector field on S. Then
the flux of $\vec{F}$ across S is

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n} \, d\sigma$$

eg 59: $S: y = x^2$ $\quad 0 \le x \le 1$, $\quad 0 \le z \le 4$



$\vec{n}$ given by the natural parametrization

$$\vec{r}(x,z) = x\hat{i} + x^2\hat{j} + z\hat{k}$$

$$\begin{cases} \vec{r}_x = \hat{i} + 2x\hat{j} \\ \vec{r}_z = \hat{k} \end{cases} \Rightarrow \vec{r}_x \times \vec{r}_z = 2x\hat{i} - \hat{j}$$

$$\therefore \vec{n} = \frac{\vec{r}_x \times \vec{r}_z}{|\vec{r}_x \times \vec{r}_z|} = \frac{2x\hat{i} - \hat{j}}{\sqrt{4x^2+1}}$$

Let $\vec{F} = yz\hat{i} + x\hat{j} - z^2\hat{k}$. Find $\iint_S \vec{F} \cdot \vec{n} \, d\sigma$.

Soln: $\displaystyle\iint_S \vec{F} \cdot \vec{n}\, d\sigma$

$$= \int_0^4 \int_0^1 \overbrace{(yz\hat{i} + x\hat{j} - z^2\hat{k})}^{\vec{F}} \cdot \overbrace{\left(\frac{2x\hat{i} - \hat{j}}{\sqrt{1+4x^2}}\right)}^{\vec{n}} \overbrace{\sqrt{1+4x^2}\, dx\, dz}^{d\sigma}$$

$$= \int_0^4 \int_0^1 (2x^3 z - x)\, dx\, dz \quad \text{(check!)}$$

$$= 2 \qquad \#$$

Remark: $$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iint_{u,v} \vec{F}(\vec{r}(u,v)) \cdot \frac{(\vec{r}_u \times \vec{r}_v)}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

$$= \iint_{u,v} \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv.$$