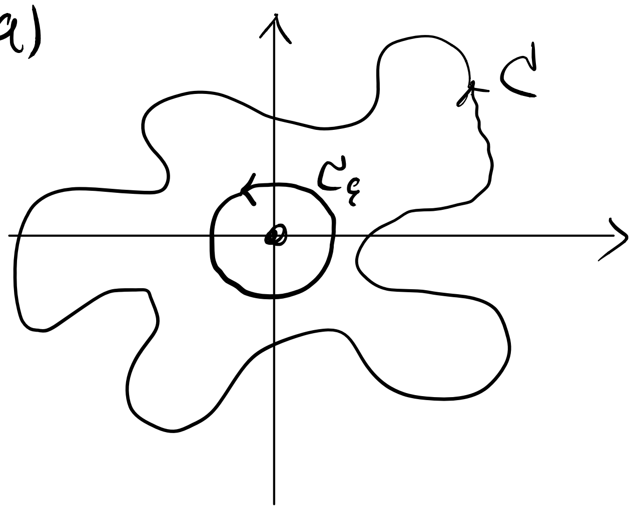
(Cont'd)

(a)



Choose $\varepsilon > 0$ small enough, such that the circle C_ε of radius ε centered at $(0,0)$ is completely enclosed by C.

By the general form of Green's Thm,

$$\oint_C \left(-\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy\right)$$

$$= \oint_{C_\varepsilon} \left(-\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy\right)$$

$\left(\text{since } \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}, \text{ where } M = -\frac{y}{x^2+y^2}, N = \frac{x}{x^2+y^2}\right)$

Parametrize C_ε by $\begin{cases} x = \varepsilon\cos\theta, \\ y = \varepsilon\sin\theta \end{cases}$, $0 \leq \theta \leq 2\pi$.

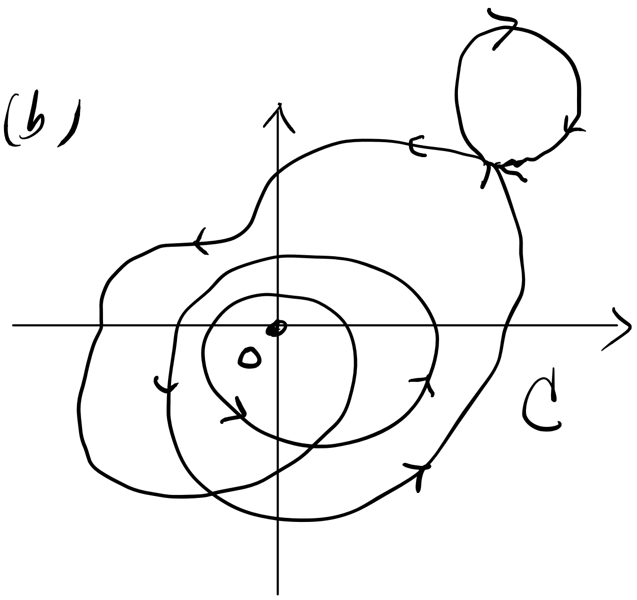
Then $\oint_{C_\varepsilon}\left(-\frac{\varepsilon\sin\theta}{\varepsilon^2}\,d(\varepsilon\cos\theta)+\frac{\varepsilon\cos\theta}{\varepsilon^2}\,d(\varepsilon\sin\theta)\right)$

$$= \int_0^{2\pi}(+\sin^2\theta+\cos^2\theta)\,d\theta = 2\pi$$

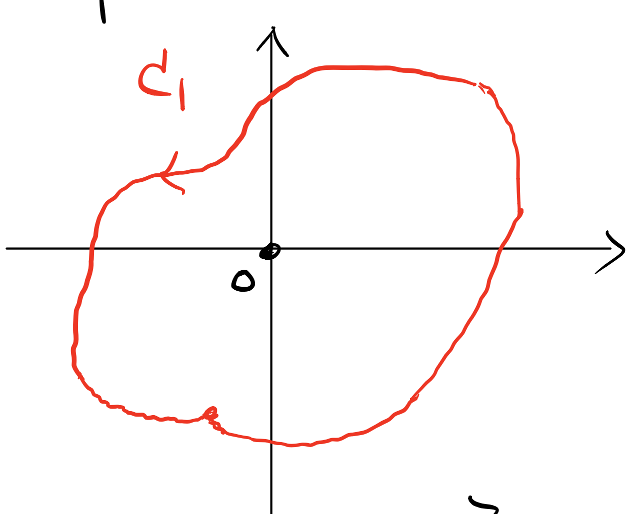
$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \oint_C (M\,dx+N\,dy) = 2\pi$$

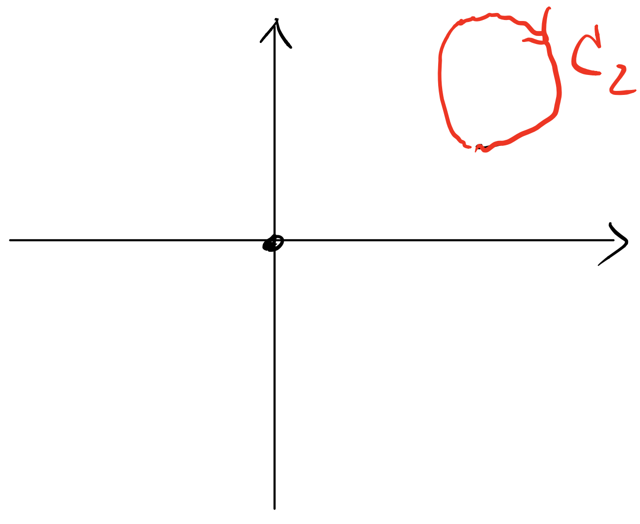
(as long as C encloses the origin.)

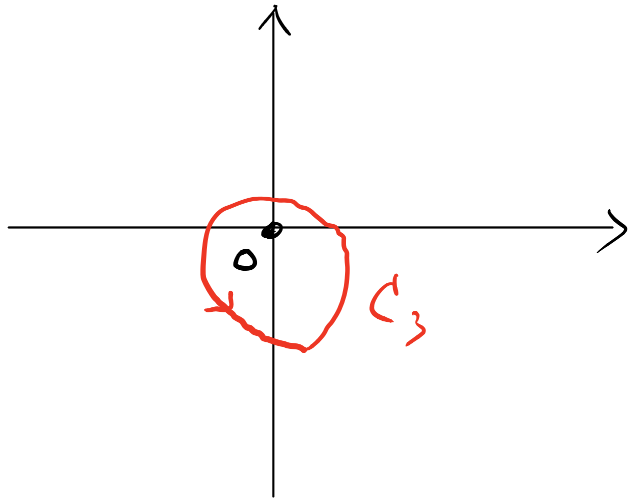
(b)

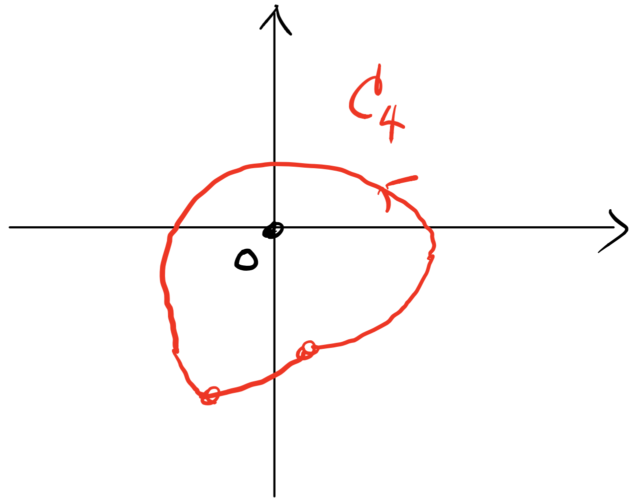


discompose the curve C into the following pieces:









Then by (a), we have

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_{C_1} \vec{F}\cdot d\vec{r} + \oint_{C_2} \vec{F}\cdot d\vec{r} + \oint_{C_3} \vec{F}\cdot d\vec{r} + \oint_{C_4} \vec{F}\cdot d\vec{r}$$

$$= 2\pi + 0 + 2\pi + 2\pi$$

$$= 6\pi$$

※

Surface Area & Integral

Def 14 Parametric Surface (Surface with parametrization)

A parametric surface (or a parametrization of a surface) in $\mathbb{R}^3$ is a mapping of 2 variables into $\mathbb{R}^3$:

$$\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}.$$

And it is call smooth if

(1) $\vec{r}$ is C^1 (i.e. $x_u, x_v, y_u, y_v, z_u, z_v$ are continuous)

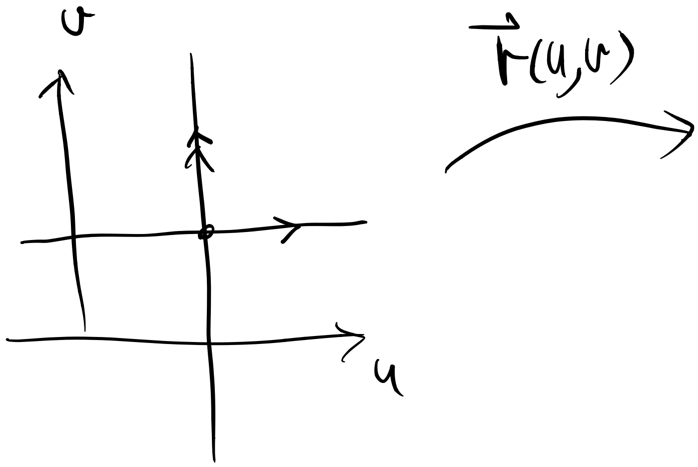
(2) $\boxed{\vec{r}_u \times \vec{r}_v \neq 0}$, $\forall u, v$

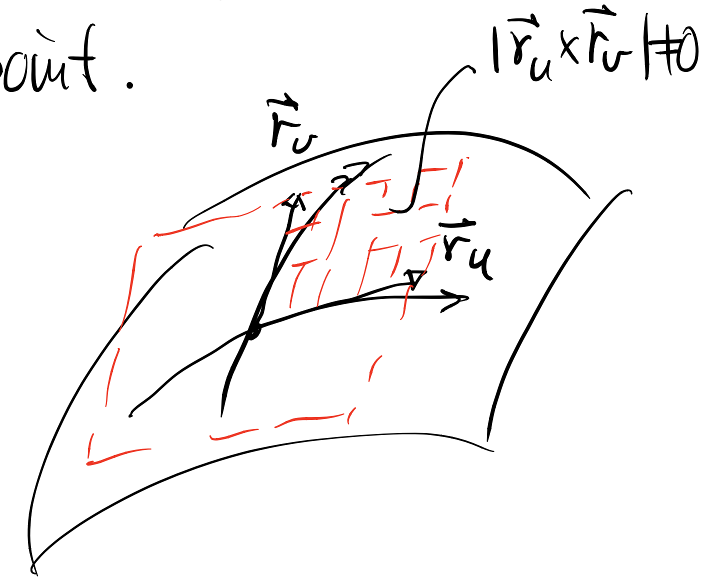
where
$$\begin{cases} \vec{r}_u = \dfrac{\partial \vec{r}}{\partial u} = \dfrac{\partial x}{\partial u}\hat{i} + \dfrac{\partial y}{\partial u}\hat{j} + \dfrac{\partial z}{\partial u}\hat{k} \\ \quad\;\; = x_u\hat{i} + y_u\hat{j} + z_u\hat{k} \\ \vec{r}_v = x_v\hat{i} + y_v\hat{j} + z_v\hat{k}. \end{cases}$$

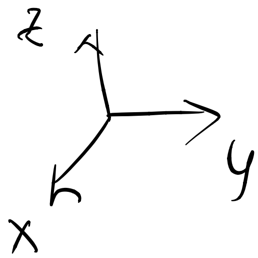
Note: Condition (2) $\Rightarrow$ $\vec{r}_u, \vec{r}_v$ are linearly independent.

$\Rightarrow$ span$(\vec{r}_u, \vec{r}_v)$ is in fact a 2-dim'l plane

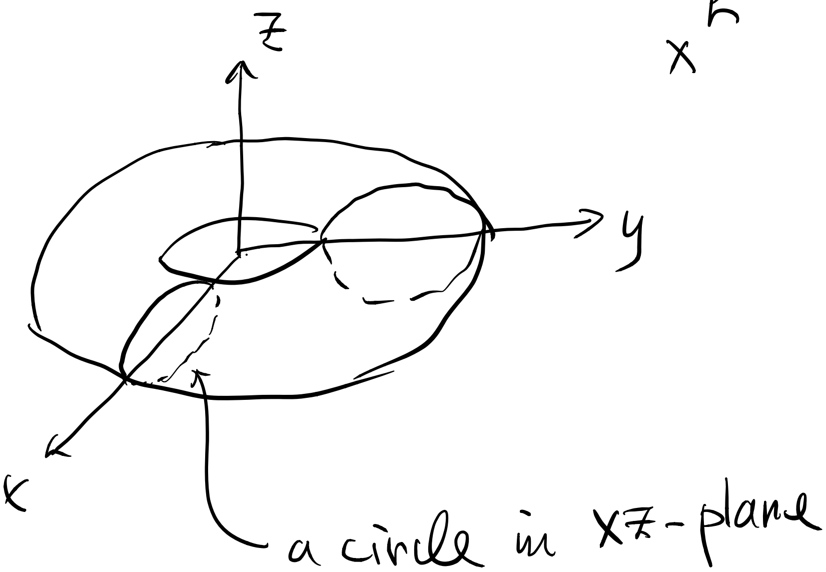
∴ "Surface" cannot be degenerated to a curve or a point.



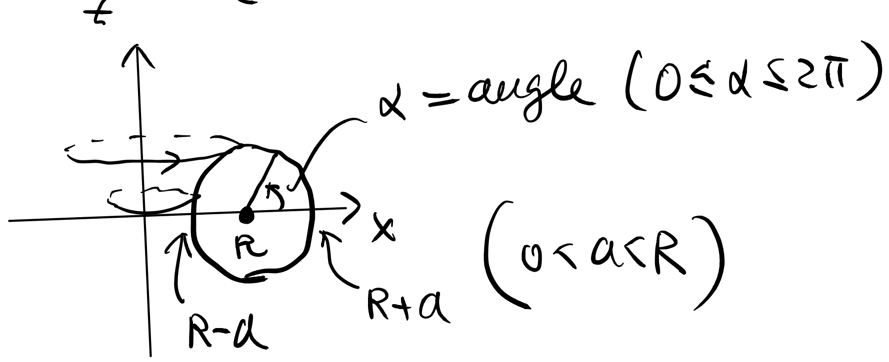




eg5!:



rotating this circle around to z-axis gives the torus.



For $y=0$ (xz-plane)
$$\begin{cases} x = R + a\cos\alpha \\ z = a\sin\alpha \end{cases}$$

Revolving around the z-axis, we have

$$\begin{cases} x = (R+a\cos\alpha)\cos\theta & 0<\alpha<2\pi \\ y = (R+a\cos\alpha)\sin\theta, & 0<\theta<2\pi \\ z = a\sin\alpha \end{cases}$$

is a parametrization of the torus.

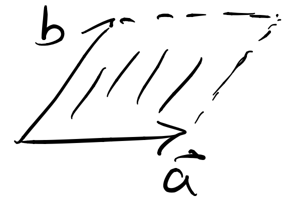
Note that this torus can also be described as

$$\left(\sqrt{x^2+y^2}-R\right)^2 + z^2 = a^2.$$

(Ex!) #

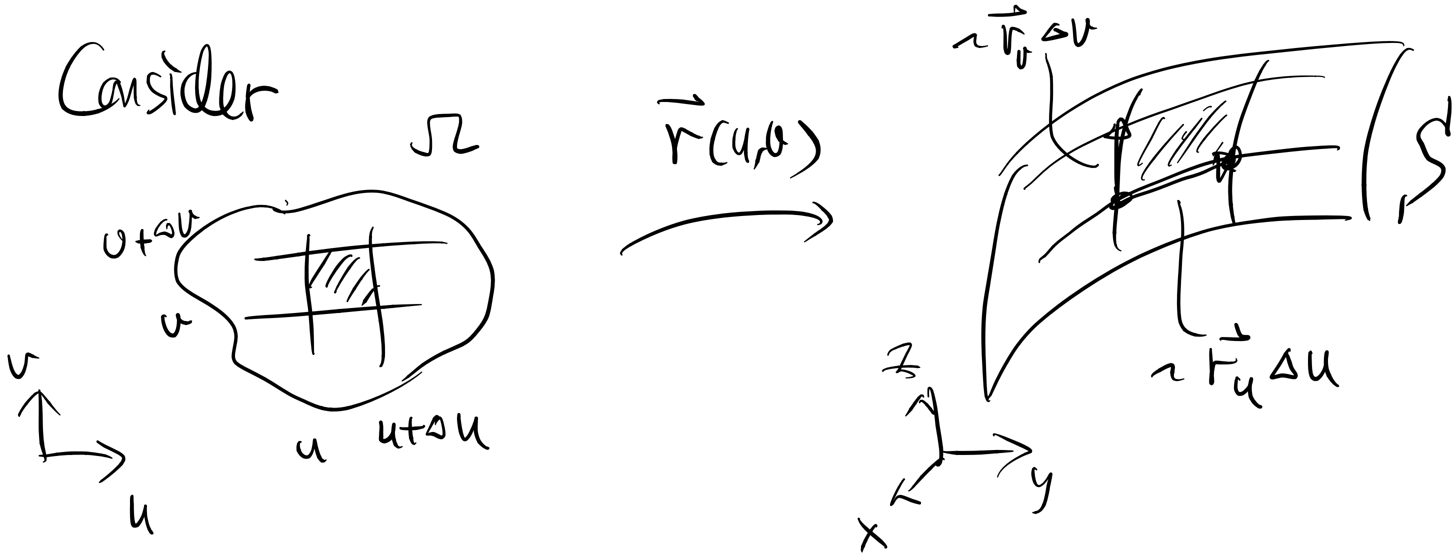
Surface Area

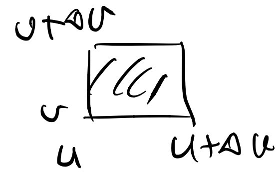
Recall: for $\vec{a}, \vec{b} \in \mathbb{R}^3$

$|\vec{a}\times\vec{b}| = $ Area ()

Let $\vec{r}(u,v)$ be a parametrization of a surface S with $(u,v)\in\Omega$.

Consider



$\Rightarrow$ "Area" on the surface corresponding to 

is $\sim \left|(\vec{r}_u \Delta u) \times (\vec{r}_v \Delta v)\right|$

$= |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$

Hence "Area element" of S denoted $d\sigma$

is given by

$$d\sigma = |\vec{r}_u \times \vec{r}_v| \, du\, dv$$

$$\boxed{d\sigma = |\vec{r}_u \times \vec{r}_v| \, dA}$$

Therefore, we make the following

Def 15: Let $S \subset \mathbb{R}^3$ be a smooth parametric surface given by $\vec{r}(u,v)$ for $(u,v) \in \Omega \subset \mathbb{R}^2$.

Then

$$\boxed{\text{Area}(S) \stackrel{\text{def}}{=} \iint_{\Omega} \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dA}$$

(i.e. $\text{Area}(S) = \iint_{\Omega} d\sigma$)

eg 32 Surface area of torus given by

$$\begin{cases} x = (R + a\cos\alpha)\cos\theta \\ y = (R + a\cos\alpha)\sin\theta \\ z = a\sin\alpha \end{cases} \qquad \begin{array}{l} 0 \le \alpha \le 2\pi \\ 0 \le \theta \le 2\pi \end{array}$$

i.e. $\vec{r}(\alpha,\theta) = (R + a\cos\alpha)\cos\theta\,\hat{i} + (R + a\cos\alpha)\sin\theta\,\hat{j} + a\sin\alpha\,\hat{k}$

$$\Rightarrow \begin{cases} \dfrac{\partial \vec{r}}{\partial \alpha} = -a\sin\alpha\cos\theta\,\hat{i} - a\sin\alpha\cos\theta\,\hat{j} + a\cos\alpha\,\hat{k} \\ \dfrac{\partial \vec{r}}{\partial \theta} = -(R + a\cos\alpha)\sin\theta\,\hat{i} + (R + a\cos\alpha)\cos\theta\,\hat{j} \end{cases}$$

$$\frac{\partial\vec{r}}{\partial\alpha}\times\frac{\partial\vec{r}}{\partial\theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin\alpha\cos\theta & -a\sin\alpha\cos\theta & a\cos\alpha \\ -(R+a\cos\alpha)\sin\theta & (R+a\cos\alpha)\cos\theta & 0 \end{vmatrix}$$

$$\overset{\text{(check)}}{=} -a(R+a\cos\alpha)\cos\theta\cos\alpha\,\hat{i}$$
$$-a(R+a\cos\alpha)\sin\theta\cos\alpha\,\hat{j}$$
$$-a\sin\alpha(R+a\cos\alpha)\,\hat{k}$$

$$\left|\frac{\partial\vec{r}}{\partial\alpha}\times\frac{\partial\vec{r}}{\partial\theta}\right| = a(R+a\cos\alpha)\left[\cos^2\theta\cos^2\alpha+\sin^2\theta\cos^2\alpha+\sin^2\alpha\right]^{1/2}$$

$$= a(R+a\cos\alpha)$$

Hence

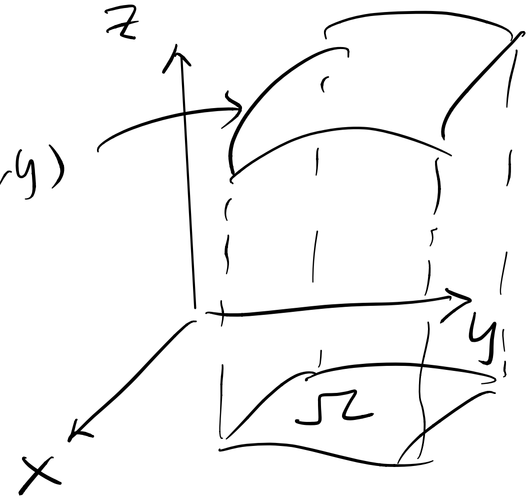
$$\text{Area(Torus)} = \iint_{\Omega}\left|\frac{\partial\vec{r}}{\partial\alpha}\times\frac{\partial\vec{r}}{\partial\theta}\right| d\alpha\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2\pi} a(R+a\cos\alpha)\, d\alpha\, d\theta$$

$$\overset{\text{(check)}}{=} 4\pi^2 Ra \quad \#$$

Surface area of a graph $z=f(x,y)$, $(x,y)\in\Omega$.

Choose the following parametrization of the graph:



$$\vec{r}(x,y) = x\hat{i} + y\hat{j} + f(x,y)\hat{k}$$

$$(x,y)\in\Omega$$

$$\Rightarrow \begin{cases} \dfrac{\partial \vec{r}}{\partial x} = \hat{i} + \dfrac{\partial f}{\partial x}\hat{k} \\ \dfrac{\partial \vec{r}}{\partial y} = \hat{j} + \dfrac{\partial f}{\partial y}\hat{k} \end{cases}$$

$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix}$$

$$= -f_x\hat{i} - f_y\hat{j} + \hat{k}$$

$$\Rightarrow \left|\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}\right| = \sqrt{1+f_x^2+f_y^2} = \sqrt{1+|\vec{\nabla} f|^2}$$

Thm 11 : The surface area of a C^1 graph S given by $z = f(x,y)$, $(x,y) \in \Omega$ is

$$\text{Area}(S) = \iint_{\Omega} \sqrt{1 + f_x^2 + f_y^2}\, dxdy$$

$$= \iint_{\Omega} \sqrt{1 + |\vec{\nabla} f|^2}\, dxdy$$

(Similarly for $x = f(y,z)$ or $y = f(x,z)$.)

Implicit Surface (Level surface)

Suppose S is given by

$$F(x,y,z) = c.$$

i.e. $S = F^{-1}(c)$.

(Note: F is a function of 3 variables, not vector field!)

eg 53 $F(x,y,z) = x^2 + y^2 + z^2$

Is $F^{-1}(0)$ a surface ?

No, since $F^{-1}(0) = \{(0,0,0)\}$ not a surface!

Remark: If $\vec{\nabla} F \neq 0$ at a point, then IFT implies that $S = F^{-1}(c)$ is a "surface" (in fact a graph) near that point.

eg53 cont'd. $\vec{\nabla} F = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\Rightarrow$ $\vec{\nabla} F = 0$ only at $(0,0,0)$.

Hence if $c>0$, then $\forall (x,y,z) \in F^{-1}(c)$, we have

$$\vec{\nabla} F(x,y,z) \neq 0.$$

$\Rightarrow$ $S = F^{-1}(c)$ $(\forall c>0)$ is a surface!

Terminology: $S = F^{-1}(c)$ is said to be smooth if (1) F is C^1 on S, and (2) $\vec{\nabla} F \neq 0$ on S.

(i.e. $S = F^{-1}(c)$ is a smooth level surface.)

How to compute surface area for a smooth level surface $S = F^{-1}(c)$?

By $\vec{\nabla} F \neq 0$, we may assume $\frac{\partial F}{\partial z} \neq 0$

(the other cases are similar)

IFT $\Rightarrow$ $S = F^{-1}(c) = \{ F(x,y,z) = c \}$

can be written as a graph $z = f(x,y)$.

(near that point)

(i.e. $F(x, y, h(x,y)) = c$)

Then chain rule $\Rightarrow$

$$\begin{cases} \dfrac{\partial f}{\partial x} = - \dfrac{F_x}{F_z} \\ \\ \dfrac{\partial f}{\partial y} = - \dfrac{F_y}{F_z} \end{cases}$$

Hence $\text{Area}(S) = \iint_\Omega \sqrt{1 + f_x^2 + f_y^2}\, dx dy$

$$= \iint_\Omega \sqrt{1 + \frac{F_x^2}{F_z^2} + \frac{F_y^2}{F_z^2}}\, dx dy$$

$$= \iint_{\Omega} \frac{\sqrt{F_z^2 + F_x^2 + F_y^2}}{|F_z|}\, dx\,dy \,.$$

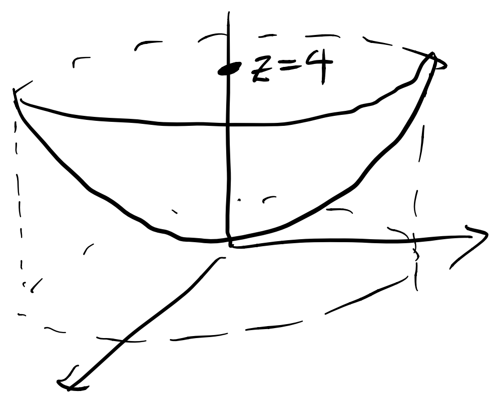
Thm 12 If $S = F^{-1}(c)$ is a smooth level surface such that $F_z \neq 0$, (and can be represented by an implicit function on a domain Ω.)

Then
$$\boxed{\text{Area}(S) = \iint_{\Omega} \frac{|\vec{\nabla} F|}{|F_z|}\, dx\,dy}$$

(Similarly for the cases that $F_x \neq 0$ & $F_y \neq 0$)

eg 54: Find surface area of the paraboloid $x^2+y^2-z=0$ below $z=4$.

(This is in fact a graph. But we do it using level surface.)



Soln: Let $F(x,y,z)$
$$= x^2+y^2-z$$

For $z=4$, $\quad x^2+y^2-z=0 \Rightarrow x^2+y^2=4$

$\Rightarrow$ Projected region $\Omega = \{(x,y) : x^2+y^2 \leq 4\}$

Check $\vec{\nabla} F = 2x\,\hat{i} + 2y\,\hat{j} - \hat{k}$

$\Rightarrow F_z = -1 \neq 0 \quad \forall (x,y) \in \Omega.$

$\therefore$ Surface Area $= \displaystyle\iint_{\Omega} \frac{|\vec{\nabla} F|}{|F_z|}\,dx\,dy$

$$= \iint_{x^2+y^2\leq 4} \frac{\sqrt{(2x)^2+(2y)^2+1^2}}{|-1|}\,dx\,dy$$

$$= \iint_{x^2+y^2\leq 4} \sqrt{1+4(x^2+y^2)}\,dx\,dy$$

check.

$$= \frac{\pi}{6}\left[(\sqrt{17})^3 - 1\right] \quad \left(\text{using polar coordinates}\right)$$

#